*Lecture Notes of the Unione Matematica Italiana* 2

Andrea Braides · Valeria Chiadò Piat (Eds.)

# Topics on Concentration Phenomena and Problems with Multiple Scales

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# Preface

The research group 'Homogenization Techniques and Asymptotic Methods for Problems with Multiple Scales', co-ordinated by Valeria Chiadò Piat and funded by INdAM-GNAMPA (*Istituto Nazionale di Alta Matematica-Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni*), operated from 2001 to 2005, involving in its activities a number of young Italian mathematicians, mainly interested in problems in the Calculus of Variations and Partial Differential Equations. One of the initiatives of that group has been the organization of a number of schools. Those in the years 2001-2003, whose lecture notes are gathered in this book, had been devoted to problems with oscillations and concentrations, while the schools in the years 2004-2005 covered a range of topics of Applied Mathematics.

The first school in Turin, 17–21 September 2001, bearing the name of the research group and devoted to problems with multiple scales, was partially disrupted by the events of September 11 of that year, one speaker, Gilles Francfort, finding himself grounded in Los Angeles, and two other speakers, Andrey Piatnitski and Gregory Chechkin, slowed down in their car trip to Italy by the tightening of the borders around the European Community. The two remaining speakers managed however to enlarge their courses to cover some extra material, encouraged by the receptive audience. The course of Andrea Braides was devoted to the description of the behaviour of variational problems on lattices as the lattice spacing tends to zero, and the various multi-scale behaviours that may be obtained from this process; that of Anneliese Defranceschi to energies with competing bulk and surface interactions. The extra lectures are not included in these notes, but some of them constitute part of the material in the book '$\Gamma$-convergence for Beginners' (Oxford U.P., 2002) by Braides. The course of Francfort, on $H$-measures, was later recovered in a 'Part II' of the School held at IAC in Rome, December 3–5, 2001, together with a contribution of Roberto Peirone on homogenization on fractals. Here we also include the text of the two courses of Piatnitski and Chechkin, while the lecture notes of the course by Francfort have appeared as a chapter of the book 'Variational Problems in Materials Science', Birkhäuser, 2006.

The second part of the present notes covers the content of the subsequent school on 'Concentration Phenomena for Variational Problems' held at the Department of Mathematics of the University of Rome 'La Sapienza', September 1–5, 2003 (co-organized by A. Braides, which explains why he appears both as an editor and as a contributor). Scope of the School was to present different problems in the Calculus of Variations depending on a small parameter $\varepsilon$, that exhibit a dramatic 'change of type' as this parameter tends to 0, that is best described by the 'concentration' of some quantity at some lower-dimensional set. The courses of Sylvia Serfaty and Didier Smets treat the case of Ginzburg-Landau energies. In a two-dimensional setting it is known that the concentration of Jacobians of minimizers at points can be interpreted as the arising of 'vortices'. A novel method envisaged by Sandier and Serfaty shows how the limit motion of these vortices can be described by making use of $\Gamma$-convergence. On the other hand, Smets's course focuses on the information that can be obtained by looking at the fine behaviour of solutions of the Allen-Cahn equations, and concerns the motion in any dimension. The use of $\Gamma$-convergence as a way to describe the concentration of maximizers of problems with sub-critical growth is also the subject of the third course by Adriana Garroni. Here the concentrating quantity is not a Jacobian (the problem is scalar), but a suitable scaling of the square of the gradient of the maximizer, that converges as a measure to a sum of Dirac masses. This phenomenon has been previously described by means of the Concentration-Compactness alternative, and this 'version' by $\Gamma$-convergence gives a new interpretation of the results.

The course of Sylvia Serfaty, originally programmed for this school, had to be postponed to a subsequent spin-off 'School on Geometric Evolution Problems' held at the Department of Mathematics of the University of Rome 'Tor Vergata', January 26–28, 2004 (with the same organizing team, and an additional course by Giovanni Bellettini) but is considered essentially part of the September 2003 School, and that is why it is included here. Other two courses, whose notes are not presented here, were held at the School by Giovanni Alberti and Halil Mete Soner. The course of Soner followed the notes of a previous school and can be found in his Lecture Notes 'Variational and dynamic problems for the Ginzburg-Landau functional. Mathematical aspects of evolving interfaces' (*Lecture Notes in Math.* **1812**, Springer, 2003, 177–233). Alberti's presentation is partly covered by his review paper 'A variational convergence result for Ginzburg-Landau functionals in any dimension' (*Boll. Un. Mat Ital.* **4** (2001), 289–310). As a final acknowledgement, it must be mentioned that these schools had been additionally jointly sponsored by the Rome and Milan Units of the National Project 'Calculus of Variations'.

Rome and Turin,
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# Contents

## Part I Problems with multiple scales

**Gamma-convergence of gradient flows and applications to Ginzburg-Landau vortex dynamics**



**PDE analysis of concentrating energies for the Ginzburg-Landau equation**

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# Part I

# Problems with multiple scales

# From discrete systems to continuous variational problems: an introduction

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## Introduction

These lecture notes cover the content of a course given by the first author at the School *Homogenization Techniques and Asymptotic Methods for Problems with Multiple Scales* in Turin, 17–21 September 2001, and previously delivered at SISSA, Trieste, jointly by both authors in a slightly enlarged version. In them, we treat the problem of the description of variational limits of discrete problems in a one-dimensional setting. Even though this is a simplified setting, the results that we are going to illustrate contain many of the features that we may encounter in a higher-dimensional framework. After this course had been held a number of papers have appeared, among which a general compactness theorem for variational limits of discrete systems in any dimension by Alicandro and Cicalese [3], accurate estimates on pointwise limits for classes of lattice systems by Blanc, Le Bris and Lions and applications to atomistic computations [10, 11, 34], an interesting analysis of the Cauchy-Born hypothesis by Friesecke and Theil [32], applications to higher-dimensional homogenization of networks by Braides and Francfort [19], to percolation problems by Braides and Piatnitski [25], to anti-phase boundaries in spin systems [1], thin discrete objects [2], etc. Not only those applications have not made the present notes obsolete, but on the contrary they have more motivated them as a necessary introduction to understand the non-trivial effects of the 'microscopic' dimension.

Given $n \in \mathbb{N}$ we consider energies of the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big( \frac{u_{i+j} - u_i}{j\lambda_n} \Big)$$

defined on $(n{+}1)$-tuples $\{u_i\}$. We may view $\{u_i\}$ as a *discrete function* defined on a lattice covering a fixed interval $[0, L]$ by introducing points $x_i^n = i\lambda_n$ ($\lambda_n = L/n$ is the *lattice spacing*). If we picture the set $\{x_i^n\}$ as the reference configuration of an array of material points interacting through some forces, and let $u_i$ represent the displacement of the $i$-th point, then $\psi_n^j$ can be thought as the energy density of the interaction of points with distance $j\lambda_n$ ($j$ lattice spacings) in the reference lattice. Note that the only assumption we make is that $\psi_n^j$ depends on $\{u_i\}$ through the differences $u_{i+j} - u_i$, but we find it more convenient to highlight its dependence on the 'discrete difference quotients'

$$\frac{u_{i+j} - u_i}{j\lambda_n}.$$

One must not be distracted from this notation and should note the generality of the approach.

Our goal is to describe the behaviour of problems of the form

$$\min\Big\{E_n(\{u_i\}) - \sum_{i=0}^{n} u_i f_i : \ u_0 = U_0,\ u_n = U_L\Big\}$$

(and similar), and to show that for a quite general class of energies these problems have a limit continuous counterpart. Here $\{f_i\}$ represents the external forces and $U_0, U_L$ are the boundary conditions at the endpoints of the interval $(0, L)$. More general statements and different problems can be also obtained. To make this asymptotic analysis precise, we use the notation and methods of $\Gamma$-convergence, for which we refer to the book by A. Braides *Γ-Convergence for Beginners* (a more complete theoretical introduction can be found in the book by G. Dal Maso *An Introduction to Γ-convergence*). We will show that, upon suitably identifying discrete functions $\{u_i\}$ with suitable (possibly discontinuous) interpolations, the free energies $E_n$ '*Γ-converge*' to a limit energy $F$. As a consequence we obtain that minimizers of the problem above are 'very close' to minimizers of

$$\min\Big\{F(u) - \int_0^L f u\, dt : \ u(0) = U_0,\ u(L) = U_L\Big\}.$$

The energies $F$ can be explicitly identified by a series of operations on the functions $\psi_n^j$. In order to give an idea of how $F$ can be described, we first consider the case when only nearest-neighbour interactions are taken into account:

$$E_n(\{u_i\}) = \sum_{i=0}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big).$$

In this case, the limit functional $F$ can be described by introducing for each $n$ a 'threshold' $T_n$ such that $T_n \to +\infty$ and $\lambda_n T_n \to 0$, and defining a limit *bulk energy density*

$$f(z) = \lim_n (\text{convex envelope of } \tilde{\psi}_n(z)),$$

and a limit *interfacial energy density*

$$g(z) = \lim_n (\text{subadditive envelope of } \lambda_n \tilde{\tilde{\psi}}_n\Big(\frac{z}{\lambda_n}\Big)),$$

where

$$\tilde{\psi}_n(z) = \begin{cases} \psi_n(z) & \text{if } |z| \le T_n \\ +\infty & \text{otherwise,} \end{cases} \qquad \tilde{\tilde{\psi}}_n(z) = \begin{cases} \psi_n(z) & \text{if } |z| \ge T_n \\ +\infty & \text{otherwise.} \end{cases}$$

Note the crucial *separation of scales* argument: essentially, the limit behaviour of $\psi_n(z)$ defines the bulk energy density, while $\lambda_n \, \psi_n(z/\lambda_n)$ determines the interfacial energy. The limit $F$ is defined (up to passing to its lower semicontinuity envelope) on piecewise-Sobolev functions as

$$F(u) = \int_{(0,L)} f(u')\, dt + \sum_{S(u)} g(u(t+) - u(t-)),$$

where $S(u)$ denotes the set of *discontinuity points* of $u$. Hence, we have a limit energy with two competing contributions of a bulk part and of an interfacial energy. In this form we can recover *fracture* and *softening* phenomena.

The description of the limit energy gets more complex when not only nearest-neighbour interactions come into play. We first examine the case when interactions up to a fixed order $K$ are taken into account:

$$E_n(\{u_i\}) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big)$$

(or, equivalently, $\psi_n^j = 0$ if $j > K$). The main idea is to show that (upon some controllable errors) we can find a lattice spacing $\eta_n$ (possibly much larger than $\lambda_n$) such that $E_n$ is 'equivalent' (as $\Gamma$-convergence is concerned) to a nearest-neighbour interaction energy on a lattice of step size $\eta_n$, of the form

$$\overline{E}_n(\{u_i\}) = \sum_{i=0}^{m-1} \eta_n \overline{\psi}_n \Big(\frac{u_{i+1} - u_i}{\eta_n}\Big),$$

and to which then the recipe above can be applied.

The crucial points here are the computation of $\overline{\psi}_n$ and the choice of the scaling $\eta_n$. In the case of *next-to-nearest neighbours* this computation is particularly simple, as it consists in choosing $\eta_n = 2\lambda_n$ and in 'integrating out the contribution of first neighbours': in formula,

$$\overline{\psi}_n(z) = \psi_n^2(z) + \frac{1}{2} \min\{\psi_n^1(z_1) + \psi_n^1(z_2) : z_1 + z_2 = 2z\}.$$

In a sense this is a formula of *relaxation type*. If $K > 2$ then the formula giving $\psi_n$ resembles more a *homogenization formula*, and we have to choose $\eta_n = K_n \lambda_n$ with $K_n$ large. In this case the reasoning that leads from $E_n$ to $\overline{E}_n$ is that the overall behaviour of a system of interacting points will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities'.

When the number of interaction orders we consider is not bounded the description becomes more complex. In particular, additional *non-local* terms may appear in $F$.

Note that *first order* $\Gamma$-limits may not capture completely the behaviour of minimizers for variational problems as above. Additional information, as *phase transitions*, *boundary layer effects* and *multiple cracking*, may be extracted from the study of *higher-order* $\Gamma$-limits.

# 1 Discrete problems with limit energies defined on Sobolev spaces

## 1.1 Discrete functionals

We will consider the limit of energies defined on one-dimensional discrete systems of $n$ points as $n$ tends to $+\infty$. In order to define a limit energy on a continuum we parameterize these points as a subset of a single interval $(0, L)$. Set

$$\lambda_n = \frac{L}{n}, \qquad x_i^n = \frac{i}{n} L = i\lambda_n, \qquad i = 0, 1, \dots, n. \tag{1}$$

We denote $I_n = \{x_0^n, \dots, x_n^n\}$ and by $\mathcal{A}_n(0, L)$ the set of functions $u : I_n \to \mathbb{R}$. If $n$ is fixed and $u \in \mathcal{A}_n(0, L)$ we equivalently denote

$$u_i = u(x_i^n).$$

Given $K \in \mathbb{N}$ with $1 \le K \le n$ and functions $f^j : \mathbb{R} \to [0, +\infty]$, with $j = 1, \dots, K$, we will consider the related functional $E : \mathcal{A}_n(0, L) \to [0, +\infty]$ given by

$$E(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} f^j(u_{i+j} - u_i). \tag{2}$$

Note that $E$ can be viewed simply as a function $E : \mathbb{R}^n \to [0, +\infty]$.

An interpretation with a physical flavour of the energy $E$ is as the internal interaction energy of a chain of $n+1$ material points each one interacting with its $K$-nearest neighbours, under the assumption that the interaction energy densities depend only on the order $j$ of the interaction and on the distance between the two points $u_{i+j} - u_i$ in the reference configuration. If $K = 1$ then each point interacts with its nearest neighbour only, while if $K = n$ then each pair of points interacts.

*Remark 1.* From elementary calculus we have that $E$ is lower semicontinuous if each $f^j$ is lower semicontinuous, and that $E$ is coercive on bounded sets of $\mathcal{A}_n(0,L)$.

### 1.2 Equivalent energies on Sobolev functions

We will describe the limit as $n \to +\infty$ of sequences $(E_n)$ with $E_n : \mathcal{A}_n(0,L) \to [0,+\infty]$ of the general form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} f_n^j(u_{i+j} - u_i). \tag{3}$$

Since each functional $E_n$ is defined on a different space, the first step is to identify each $\mathcal{A}_n(0,L)$ with a subspace of a common space of functions defined on $(0,L)$. In order to identify each discrete function with a continuous counterpart, we extend $u$ by $\tilde{u} : (0,L) \to \mathbb{R}$ as the piecewise-affine function defined by

$$\tilde{u}(s) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda_n}(s - x_{i-1}) \qquad \text{if } s \in (x_{i-1}, x_i). \tag{4}$$

In this case, $\mathcal{A}_n(0,L)$ is identified with those continuous $u \in W^{1,1}(0,L)$ (actually, in $W^{1,\infty}(0,L)$) such that $u$ is affine on each interval $(x_{i-1}, x_i)$. Note moreover that we have

$$\tilde{u}' = \frac{u_i - u_{i-1}}{\lambda_n} \tag{5}$$

on $(x_{i-1}, x_i)$. If no confusion is possible, we will simply write $u$ in place of $\tilde{u}_n$.

It is convenient to rewrite the dependence of the energy densities in (3) with respect to difference quotients rather than the differences $u_{i+j} - u_i$. We then write

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{6}$$

where

$$\psi_n^j(z) = \frac{1}{\lambda_n} f_n^j(j\lambda_n z).$$

With the identification of $u$ with $\tilde{u}$, $E_n$ may be viewed as an integral functional defined on $W^{1,1}(0,L)$. In fact, for fixed $j \in \{0, \dots, K-1\}$, $k \in \{0, \dots, n-1\}$ and $i$ such that $i \le k < i+j$ we have

$$\frac{u_{i+j} - u_i}{j\lambda_n} = \frac{1}{j} \sum_{m=i-k}^{i-k+j-1} \frac{u_{k+m+1} - u_{k+m}}{\lambda_n} = \frac{1}{j} \sum_{m=i-k}^{i-k+j-1} \tilde{u}'(x + m\lambda_n)$$

for all $x \in (x_k^n, x_{k+1}^n)$, so that

$$\lambda_n \psi^j\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big) = \frac{1}{j}\sum_{k=i}^{i+j-1}\int_{x^n_k}^{x^n_{k+1}} \psi^j\Big(\frac{1}{j}\sum_{m=i-k}^{i-k+j-1}\tilde u'(x+m\lambda_n)\Big)\,dx.$$

We then get

$$\sum_{i=0}^{n-j}\lambda_n \psi^j_n\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big) = \frac{1}{j}\sum_{l=0}^{j-1}\int_{l\lambda_n}^{L-(j-1-l)\lambda_n} \psi^j_n\Big(\frac{1}{j}\sum_{k=-l}^{j-1-l}\tilde u'(x+k\lambda_n)\Big)dx.$$

and the equality

$$E_n(u) = F_n(\tilde u), \tag{7}$$

where

$$F_n(v) = \begin{cases} \displaystyle\sum_{j=1}^{K_n}\sum_{l=0}^{j-1}\frac{1}{j}\int_{l\lambda_n}^{L-(j-1-l)\lambda_n} \psi^j_n\Big(\frac{1}{j}\sum_{k=-l}^{j-1-l} v'(x+k\lambda_n)\Big)dx & \text{if } v\in\mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.}\end{cases} \tag{8}$$

Note that in the particular case $K_n = 1$ we have (set $\psi_n = \psi^1_n$)

$$F_n(v) = \begin{cases} \displaystyle\int_0^L \psi_n(v')dx & \text{if } v\in\mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.}\end{cases} \tag{9}$$

**Definition 1 (Convergence of discrete functions and energies).** With the identifications above we will say that $u_n$ *converges* to $u$ (respectively, in $L^1$, in measure, in $W^{1,1}$, etc.) if $\tilde u_n$ converge to $u$ (respectively, in $L^1$, in measure, weakly in $W^{1,1}$, etc.), and we will say that $E_n$ *$\Gamma$-converges* to $F$ (respectively, with respect to the convergence in $L^1$, in measure, weakly in $W^{1,1}$, etc.) if $F_n$ $\Gamma$-converges to $F$ (respectively, with respect to the convergence in $L^1$, in measure, weakly in $W^{1,1}$, etc.). We refer to [14] for the notation and main definitions of $\Gamma$-convergence.

### 1.3 Convex energies

We first treat the case when the energies $\psi^j_n$ are convex. We will see that in the case of nearest neighbours, the limit is obtained by simply replacing sums by integrals, while in the case of long-range interactions a superposition principle holds.

For simplicity we suppose that the energy densities do not depend on $n$; *i.e.*,

$$\psi^j_n = \psi^j.$$

**Nearest-neighbour interactions**

We start by considering the case $K = 1$, so that the functionals $E_n$ are given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big). \tag{10}$$

The integral counterpart of $E_n$ is given by

$$F_n(v) = \begin{cases} \displaystyle\int_0^L \psi(v')dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

Note that $F_n$ depends on $n$ only through its domain $\mathcal{A}_n(0,L)$.

The following result states that as $n$ approaches $\infty$ the identification of $E_n$ with its continuous analogue is complete.

**Theorem 1.** *Let $\psi : \mathbb{R} \to [0,+\infty)$ be convex and let $E_n$ be given by* (10)*.*

(i) *The $\Gamma$-limit of $E_n$ with respect to the weak convergence in $W^{1,1}(0,L)$ is given by $F$ defined by*

$$F(u) = \int_{(0,L)} \psi(u')\,dx. \tag{12}$$

(ii) *If*

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty \tag{13}$$

*then the $\Gamma$-limit of $E_n$ with respect to the convergence in $L^1(0,L)$ is given by $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \textit{if } u \in W^{1,1}(0,L) \\ +\infty & \textit{otherwise} \end{cases} \tag{14}$$

*on $L^1(0,L)$.*

*Proof.* (i) The functional $F$ defines a weakly lower semicontinuous functional on $W^{1,1}(0,L)$ and clearly $F_n \geq F$; hence also we have $\Gamma$-$\liminf_j F_j(u) \geq F(u)$. Conversely, fixed $u \in W^{1,1}(0,L)$ let $u_n \in \mathcal{A}_n(0,L)$ be such that $u_n(x_i^n) = u(x_i^n)$. By convexity we have

$$\int_{x_i^n}^{x_{i+1}^n} \psi(u')\,dt \geq \lambda_n \psi\Big(\frac{1}{\lambda_n}\int_{x_i^n}^{x_{i+1}^n} u'\,dt\Big) = \lambda_n \psi\Big(\frac{u(x_{i+1}^n) - u(x_i^n)}{\lambda_n}\Big);$$

hence, summing up,

$$\int_0^L \psi(u')\,dt \geq E_n(u_n).$$

This shows that $(u_n)$ is a recovery sequence for $F$.

(ii) If (13) holds then the sequence $(E_n)$ is equi-coercive on bounded sets of $L^1(0,L)$ with respect to the weak convergence in $W^{1,1}(0,L)$, from which the thesis is easily deduced. □

### Long-range interactions

Let now $K \in N$ be fixed. The energies $E_n$ take the form

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big( \frac{u_{i+j} - u_i}{j\lambda_n} \Big). \tag{15}$$

**Theorem 2.** *Let $\psi^j : \mathbb{R} \to [0,+\infty)$ be convex and let $E_n$ be given by* (15)*. Let $\psi^1$ satisfy*

$$\lim_{|z|\to\infty} \frac{\psi^1(z)}{|z|} = +\infty \tag{16}$$

*then the $\Gamma$-limit of $E_n$ with respect to the convergence in $L^1(0,L)$ is given by $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in W^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{17}$$

*on $L^1(0,L)$, where*

$$\psi = \sum_{j=1}^{K} \psi^j. \tag{18}$$

*Proof.* Note that $(E_n)$ is equi-coercive on bounded set of $L^1(0,L)$ as in the proof of Theorem 1. Then it suffices to check the $\Gamma$-limit on $W^{1,1}(0,L)$.

To prove the $\Gamma$-liminf inequality let $u_n \rightharpoonup u$ weakly in $W^{1,1}(0,L)$. Then, for every $j \in \{0,\ldots,K\}$ and $l \in \{0,\ldots,j-1\}$, also the convex combination

$$u_n^{j,l} = \frac{1}{j} \sum_{k=-l}^{j-1-l} \tilde{u}_n(x + k\lambda_n)$$

converge weakly to $u$ in $W^{1,1}_{\rm loc}(0,L)$. By (7) then we have, for all fixed $\eta > 0$,

$$\begin{aligned} \liminf_n E_n(u) &\geq \liminf_n \sum_{j=1}^{K} \frac{1}{j} \sum_{l=0}^{j-1} \int_\eta^{L-\eta} \psi_n^j \Big( (u_n^{j,l})' \Big) dx \\ &\geq \sum_{j=1}^{K} \sum_{l=0}^{j-1} \frac{1}{j} \liminf_n \int_\eta^{L-\eta} \psi_n^j \Big( (u_n^{j,l})' \Big) dx \\ &\geq \sum_{j=1}^{K} \sum_{l=0}^{j-1} \frac{1}{j} \int_\eta^{L-\eta} \psi^j(u') dx = \int_\eta^{L-\eta} \psi(u')\,dt. \end{aligned}$$

The liminf inequality follows by the arbitrariness of $\eta$.

Again, fixed $u \in W^{1,1}(0,L)$ let $u_n \in \mathcal{A}_n(0,L)$ be such that $u_n(x_i^n) = u(x_i^n)$. By Jensen's inequality,

$$E_n(u_n) = \sum_{j=1}^{K}\sum_{i=0}^{n-j}\lambda_n\psi^j\Big(\frac{1}{j\lambda_n}\int_{x_i^n}^{x_{i+j}^n} u'\,dt\Big) \le \sum_{j=1}^{K}\sum_{i=0}^{n-j}\frac{1}{j}\int_{x_i^n}^{x_{i+j}^n}\psi^j(u')dt$$
$$= \sum_{j=1}^{K}\frac{1}{j}\sum_{i=0}^{n-j}\int_{x_i^n}^{x_{i+j}^n}\psi^j(u')dt \le \sum_{j=1}^{K}\int_0^L \psi^j(u')dt,$$

which implies the limsup inequality. □

## 1.4 Non-convex energies with superlinear growth

We now investigate the effects of the lack of convexity, always in the framework of limits defined on Sobolev spaces. Again we suppose that the energy densities do not depend on $n$; *i.e.*,

$$\psi_n^j = \psi^j,$$

but are not necessarily convex.

### Nearest-neighbour interactions

We consider the case $K = 1$. In this case the only effect of the passage from the discrete setting to the continuum is a convexification of the integrand.

**Theorem 3.** *Let $\psi : \mathbb{R} \to [0,+\infty)$ be a Borel function satisfying* (13). *Let $E_n$ be given by* (10)*; then the $\Gamma$-limit of $E_n$ with respect to the convergence in $L^1(0,L)$ is given by $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi^{**}(u')\,dx & \text{if } u \in W^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{19}$$

*on $L^1(0,L)$.*

*Proof.* The $\Gamma$-liminf inequality immediately follows as in the proof of Theorem 1(i).

As for the limsup inequality, first note that if $u \in W^{1,1}(0,L)$ and $\psi(u') = \psi^{**}(u')$ a.e. then we may simply take $u_n$ as in the proof of Theorem 1(i), so that for such $u$ we have $\Gamma$-$\lim_n E_n(u) = F(u)$. If $\psi$ is lower semicontinuous and $u$ is affine with $u' = z$, let $z_1, z_2 \in \mathbb{R}$ and $\lambda \in [0,1]$ be such that

$$z = \lambda z_1 + (1-\lambda)z_2, \qquad \psi(z_1) = \psi^{**}(z_1), \psi(z_2) = \psi^{**}(z_2)$$

and

$$\psi^{**}(z) = \lambda\psi(z_1) + (1-\lambda)\psi(z_2).$$

Then there exists $u_j$ weakly converging to $u$ such that $u_j' \in \{z_1, z_2\}$ and $F(u) = \lim_j F(u_j)$. By the lower semicontinuity of the $\Gamma$-limsup we then have

$$\Gamma\text{-}\limsup_n E_n(u) \le \liminf_j \Gamma\text{-}\limsup_n E_n(u_j) = \liminf_j F(u_j) = F(u),$$

as desired. If $\psi$ is not lower semicontinuous then suitable $z_{1,j}$ and $z_{2,j}$ must be chosen such that $u_j$ weakly converges to $u$, $u_j' \in \{z_{1,j}, z_{2,j}\}$ and $F(u) = \lim_j F(u_j)$.

To conclude the proof it remains to suitably approximate any function $u \in W^{1,1}(0,L)$ by some its affine interpolations $(u_k)$ and remark that by the convexity of $F$ we have $F(u) = \lim_k F(u_k)$. $\square$

**Next-to-nearest neighbour interactions**

In the non-convex setting, the case $K = 2$ offers an interesting way of describing the two-level interactions between first and second neighbours. Such description is more difficult in the case $K \ge 3$. Essentially, the way the limit continuum theory is obtained is by first integrating-out the contribution due to nearest neighbours by means of an inf-convolution procedure and then by applying the previous results to the resulting functional.

**Theorem 4.** *Let $\psi^1, \psi^2 : \mathbb{R} \to [0, +\infty)$ be Borel functions such that*

$$\lim_{|z|\to\infty} \frac{\psi^1(z)}{|z|} = +\infty, \tag{20}$$

*and let $E_n(u) : \mathcal{A}_n(0,L) \to [0,+\infty)$ be given by*

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi^1\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) \tag{21}$$

*Let $\tilde\psi : \mathbb{R} \to [0,+\infty)$ be defined by*

$$\begin{aligned}\tilde\psi(z) &= \psi^2(z) + \frac{1}{2}\inf\{\psi^1(z_1) + \psi^1(z_2)) : z_1 + z_2 = 2z\} \\ &= \inf\Big\{\psi^2(z) + \frac{1}{2}(\psi^1(z_1) + \psi^1(z_2)) : z_1 + z_2 = 2z\Big\},\end{aligned} \tag{22}$$

*and let*

$$\psi = \tilde\psi^{**}. \tag{23}$$

*Then the $\Gamma$-limit of $E_n$ with respect to the convergence in $L^1(0,L)$ is given by $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in W^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{24}$$

*on $L^1(0,L)$.*

*Remark 2.* (i) The growth conditions on $\psi^2$ can be weakened, by requiring that $\psi^2 : \mathbb{R} \to \mathbb{R}$ and

$$-c_1\psi^1 \le \psi^2 \le c_2(1+\psi^1)$$

provided that we still have

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty.$$

(ii) If $\psi^1$ is convex then $\tilde{\psi} = \psi^1 + \psi^2$. If also $\psi^2$ is convex then we recover a particular case of Theorem 2.

*Proof.* Let $u \in \mathcal{A}_n(0,L)$. We have, regrouping the terms in the summation,

$$\begin{aligned} E_n(u) &= \sum_{\substack{i=0\\ i \text{ even}}}^{n-2} \lambda_n\Big(\psi^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\frac{1}{2}\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)+\frac{1}{2}\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)\Big)\\ &\quad+\sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} \lambda_n\Big(\psi^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\frac{1}{2}\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)+\frac{1}{2}\psi^1\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)\Big)\\ &\quad+\frac{\lambda_n}{2}\psi^1\Big(\frac{u_n-u_{n-1}}{\lambda_n}\Big)+\frac{1}{2}\psi^1\Big(\frac{u_1-u_0}{\lambda_n}\Big)\\ &\ge \frac{1}{2}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-2} 2\lambda_n\tilde{\psi}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde{\psi}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right)\\ &\ge \frac{1}{2}\left(\sum_{\substack{i=0\\ i \text{ even}}}^{n-2} 2\lambda_n\psi\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)+\sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} 2\lambda_n\psi\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right)\\ &= \frac{1}{2}\Big(\int_0^{2\lambda_n[n/2]}\psi(\tilde{u}_1')\,dt+\int_{\lambda_n}^{(1+2[n-1/2])\lambda_n}\psi(\tilde{u}_2')\,dt\Big), \end{aligned} \tag{25}$$

where $\tilde{u}_k$, respectively, with $k = 1, 2$, are the continuous piecewise-affine functions such that

$$\tilde{u}_k' = \frac{u_{i+2}-u_i}{2\lambda_n} \qquad \text{on } (x_i^n, x_{i+2}^n) \tag{26}$$

for $i$, respectively, even or odd.

Let now $u_n \to u$ in $L^1(0,L)$ and $\sup_n E_n(u_n) < +\infty$; then $u_n \rightharpoonup u$ in $W^{1,1}(0,L)$. Let $u_{k,n}$ be defined as in (26); as in the proof of Theorem 2, we deduce $u_{k,n} \rightharpoonup u$ as $n \to +\infty$, for $k = 1, 2$. For every fixed $\eta > 0$ by (25) we obtain

$$\begin{aligned} \liminf_n E_n(u_n) &\ge \frac{1}{2}\left(\liminf_n \int_\eta^{L-\eta}\psi(u_{1,n}')\,dt+\liminf_n\int_\eta^{L-\eta}\psi(u_{2,n}')\,dt\right)\\ &\ge \int_\eta^{L-\eta}\psi(u')\,dt, \end{aligned}$$

and the liminf inequality follows by the arbitrariness of $\eta > 0$.

Now we prove the limsup inequality. By arguing as in the proof of Theorem 3, note that it suffices to treat the case when $\tilde{\psi}$ is lower semicontinuous, $u(x) = zx$ and $\psi(z) = \tilde{\psi}(z)$. With fixed $\eta > 0$ let $z_1$, $z_2$ be such that $z_1 + z_2 = 2z$ and

$$\psi^2(z) + \frac{1}{2}(\psi^1(z_1) + \psi^2(z_2)) \leq \tilde{\psi}(z) + \eta.$$

We define the recovery sequence $u_n$ as

$$u_n(x_i^n) = \begin{cases} zx_i^n & \text{if } i \text{ is even} \\ z(i-1)\lambda_n + z_1\lambda_n & \text{if } i \text{ is odd.} \end{cases}$$

We then have

$$\begin{aligned} E_n(u_n) &= \sum_{i=0}^{n-1} \lambda_n \psi^1\Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi^2\Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n}\Big) \\ &\leq \frac{L}{2}(\psi^1(z_1) + \psi^1(z_2)) + L\psi^2(z) \leq L\tilde{\psi}(z) = L\psi(z) = F(u) \end{aligned}$$

as desired. □

*Remark 3 (Multiple-scale effects).* The formula defining $\psi$ highlights a double-scale effect. The operation of inf-convolution highlights oscillations on the scale $\lambda_n$, while the convexification of $\tilde{\psi}$ acts at a much larger scale.

**Long-range interactions**

We consider now the case of a general $K \geq 1$. In this case the effective energy density will be given by a homogenization formula. We suppose for the sake of simplicity that $\psi^j : \mathbb{R} \to [0, +\infty)$ are lower semicontinuous and there exists $p > 1$ such that

$$\psi^1(z) \geq c_0(|z|^p - 1), \qquad \psi^j(z) \leq c_j(1 + |z|^p). \tag{27}$$

for all $j = 1, \ldots, K$. Before stating the convergence result we define some energy densities.

Let $N \in \mathbb{N}$. We define $\psi_N : \mathbb{R} \to [0, +\infty)$ as follows:

$$\begin{aligned} \psi_N(z) = \min\Big\{ &\frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j\Big(\frac{u(i+j) - u(i)}{j}\Big) \\ &u : \{0, \ldots, N\} \to \mathbb{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K \Big\}. \end{aligned} \tag{28}$$

**Proposition 1.** *For all $z \in \mathbb{R}$ there exists the limit $\psi(z) = \lim_N \psi_N(z)$.*

*Proof.* With fixed $z \in \mathbb{R}$, let $N, M \in \mathbb{N}$ with $M > N$, and let $u_N$ be a minimizer for $\psi_N(z)$. We define $u_M : \{0, \dots, M\} \to \mathbb{R}$ as follows:

$$u_M(i) = \begin{cases} u_N(i - lN) + lNz & \text{if } lN \le i \le (l+1)N \ (0 \le l \le \frac{M}{N} - 1) \\ zi & \text{otherwise.} \end{cases}$$

Then we can estimate

$$\begin{aligned} \psi_M(z) &\le \frac{1}{M} \sum_{j=1}^{K} \sum_{i=0}^{M-j} \psi^j \Big( \frac{u_M(i+j) - u_M(i)}{j} \Big) \\ &\le \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j \Big( \frac{u_N(i+j) - u_N(i)}{j} \Big) \\ &\quad + \frac{1}{N} \sum_{j=1}^{K} (2K - j) \psi^j(z) + \sum_{j=1}^{K} \frac{M - [M/N]N + k - j}{M} \psi^j(z) \\ &\le \psi_N(z) + \frac{2K}{N} \sum_{j=1}^{K} \psi_j(z) + \frac{N+K}{M} \sum_{j=1}^{K} \psi_j(z) \\ &\le \psi_N(z) + c \Big( \frac{2K}{N} + \frac{N+K}{M} \Big) (1 + |z|^p). \end{aligned} \tag{29}$$

Taking first the limsup in $M$ and then the liminf in $N$ we deduce that

$$\limsup_M \psi_M(z) \le \liminf_N \psi_N(z)$$

as desired. □

*Remark 4.* The following properties can be easily proved:
(i) $c_0(|z|^p - 1) \le \psi^1(z) \le \psi(z) \le c(1 + |z|^p)$;
(ii) $\psi$ is lower semicontinuous;
(iii) $\psi$ is convex;
(iv) for all $N \in \mathbb{N}$ we have $\psi(z) \le \psi_N(z) + \frac{c}{N}(1 + |z|^p)$.

We can state the convergence theorem.

**Theorem 5.** *Let $\psi^j$ be as above and let $E_n$ be defined by* (15)*. Then the $\Gamma$-limit of $E_n$ with respect to the convergence in $L^1(0, L)$ is given by $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u') \, dx & \text{if } u \in W^{1,p}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{30}$$

*on $L^1(0, L)$, where $\psi$ is given by Proposition* 1.

*Proof.* We begin by establishing the liminf inequality. Let $u_n \to u$ in $L^1(0,L)$ be such that $\sup_n E_n(u_n) < +\infty$. Note that this implies that

$$\sup_n \int_0^L |u_n'|^p\, dt < +\infty,$$

so that indeed $u_n \rightharpoonup u$ weakly in $W^{1,p}(0,L)$ and hence also $u_n \to u$ in $L^\infty(0,L)$.

For all $k \in \{0,\dots,N-1\}$ let

$$\Phi_n(k) = \sum_{l\in\mathbb{N}} \int_{((k+Nl-2K)\lambda_n,(k+Nl+2K)\lambda_n)\cap(0,L)} |u_n'|^p\, dt.$$

We have

$$\sum_{k=0}^{N-1} \Phi_n(k) \le 2K \int_0^L |u_n'|^p\, dt \le c,$$

so that, upon choosing a subsequence if necessary, there exists $k$ such that

$$\Phi_n(k) \le \frac{c}{N}.$$

For the sake of notational simplicity we will suppose that this holds with $k=0$, and also that $n = MN$ with $M \in \mathbb{N}$, so that the inequality above reads

$$\sum_{l=0}^{M-1} \int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)\cap(0,L)} |u_n'|^p\, dt. \tag{31}$$

We may always suppose so, upon first reasoning in slightly smaller intervals than $(0,L)$ and then let those intervals invade $(0,L)$.

Let $v_n^N$ be the piecewise-affine function defined on $(0,L)$ such that

$$\begin{aligned}
&v_n^N(0) = u_n(0)\\
&(v_n^N)' = u_n' \qquad \text{on } (x_i^n, x_{i+1}^n),\ Nl+K \le i \le N(l+1)-K-1\\
&(v_n^N)' = \frac{u_n((Nl+N-K)\lambda_n) - u_n((Nl+K)\lambda_n)}{(N-2K)\lambda_n} =: z_{n,l}^N\\
&\qquad\qquad \text{on } (Nl\lambda_n,(Nl+K)\lambda_n) \cup ((N(l+1)-K)\lambda_n, N(l+1)\lambda_n).
\end{aligned}$$

The construction of $v_n^N$ deserves some words of explanation. The function $v_n^N$ is constructed on each interval $(Nl\lambda_n, N(l+1)\lambda_n)$ as equal to the function $u_n$ (up to an additive constant) in the middle interval $((Nl+K)\lambda_n, (N(l+1)-K)\lambda_n)$, and as the affine function of slope $z_{n,l}^N$ in the remaining two intervals. Note that the construction implies that the function

$$v_{n,l}^N : \{0,\dots,N\} \to \mathbb{R}$$

defined by

$$v_{n,l}^N(i) = \frac{1}{\lambda_n} v_n^N((lN+i)\lambda_n)$$

is a test function for the minimum problem defining $\psi_N(z_{n,l}^N)$, and that

$$\begin{aligned}
&\sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big( \frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) \\
&= \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big( \frac{v_{n,l}^N((i+j)) - v_{n,l}^N(i)}{j} \Big) \geq N\lambda_n \psi_N(z_{n,l}^N). \qquad (32)
\end{aligned}$$

Note moreover that, by Hölder's inequality, we have

$$\int_{(0,L)} |(v_n^N)' - u_n'| \, dt \leq \Big( \frac{2K}{N} L \Big)^{1-1/p} \|u_n'\|_{L^p(0,L)} + \frac{2K}{N-2K} \|u_n'\|_{L^1(0,L)},$$

so that, since $u_n(0) = v_n^N(0)$ we have a uniform bound

$$\|v_n^N - u_n\|_{L^\infty(0,L)} \leq \frac{C}{N}. \qquad (33)$$

We have that

$$\begin{aligned}
E_n(u_n) &\geq \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl+K}^{N(l+1)-K-j} \lambda_n \psi^j \Big( \frac{u_n(x_{i+j}^n) - u_n(x_i^n)}{j\lambda_n} \Big) \\
&= \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl+K}^{N(l+1)-K-j} \lambda_n \psi^j \Big( \frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) \\
&= \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big( \frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) \\
&\qquad - \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl}^{Nl+K} \lambda_n \psi^j \Big( \frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) \\
&\qquad - \sum_{l=1}^{M} \sum_{j=1}^{K} \sum_{i=Nl-K-j}^{Nl-j} \lambda_n \psi^j \Big( \frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) \\
&=: \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big( \frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big) - I_n^1 - I_n^2 \\
&\geq \sum_{l=0}^{M-1} \sum_{j=1}^{K} N\lambda_n \psi_N(z_{n,l}^N) - I_n^1 - I_n^2, \qquad (34)
\end{aligned}$$

the last estimate being given by (32).

We give an estimate of the term $I_n^1$; the term $I_n^2$ can be dealt with similarly. Let $i < Nl + K \le i + j$; by the growth conditions on $\psi^j$ and the convexity of $z \mapsto |z|^p$ we have

$$\psi^j\Big(\frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n}\Big)$$
$$\le c\Big(1 + \Big|\frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n}\Big|^p\Big) \le c\Big(1 + \frac{1}{j}\sum_{k=i}^{i+j-1}\Big|\frac{v_n^N(x_{i+1}^n) - v_n^N(x_i^n)}{\lambda_n}\Big|^p\Big)$$
$$\le c\Big(1 + K|z_{n,l}^N|^p + \frac{1}{\lambda_n}\int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)\cap(0,L)} |u_n'|^p\,dt\Big)$$

We then deduce by (31) and the fact that

$$|z|^p \le c(1 + \psi_N(z))$$

that

$$I_n^1 \le \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{Nl+K} \lambda_n c\Big(1 + \psi_N(z_{n,l}^N) + \frac{1}{\lambda_n}\int_{((Nl+K)\lambda_n,(Nl+2K)\lambda_n)} |u_n'|^p\,dt\Big)$$
$$\le \frac{c}{N} + \frac{c}{N}\sum_{l=0}^{M-1} N\lambda_n\psi_N(z_{n,l}^N). \tag{35}$$

Plugging this estimate and the analogue for $I_n^2$ into (34) we get

$$E_n(u_n) \ge \Big(1 - \frac{c}{N}\Big)\sum_{l=0}^{M-1} N\lambda_n\psi_N(z_{n,l}^N) - \frac{c}{N}. \tag{36}$$

By Remark 4(iv) we have

$$\psi_N(z) \ge \psi(z) - \frac{c}{N}(1 + |z|^p) \ge \Big(1 - \frac{c}{N}\Big)\psi(z) - \frac{c}{N}.$$

From (36) we then have

$$E_n(u_n) \ge \Big(1 - \frac{c}{N}\Big)\sum_{l=0}^{M-1} N\lambda_n\psi(z_{n,l}^N) - \frac{c}{N}.$$

Now, note that the piecewise-affine functions $u_n^N$ defined by

$$u_n^N(0) = u_n(0) \qquad \text{and} \qquad (u_n^N)' = z_{n,l}^N \text{ on } (Nl\lambda_n, N(l+1)\lambda_n)$$

are weakly precompact in $W^{1,p}(0,L)$, so that we may suppose that $u_n^N \rightharpoonup u^N$. Then by Theorem 1 we have

$$\liminf_n \sum_{l=0}^{M-1} N\lambda_n \psi(z^N_{n,l}) = \liminf_n \int_0^L \psi((u^N_n)')\,dt \geq \int_0^L \psi((u^N)')\,dt, \quad (37)$$

so that

$$\liminf_n E_n(u_n) \geq \Big(1-\frac{c}{N}\Big)\int_0^L \psi((u^N)')\,dt - \frac{c}{N}.$$

By (33) and the uniform convergence of $u_n$ to $u$ we have

$$\|u^N - u\|_{L^\infty(0,L)} \leq \frac{c}{N}. \quad (38)$$

By letting $N \to +\infty$ we then obtain the thesis by the lower semicontinuity of $\int \psi(u')\,dt$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such $u$ it suffices to fix $N \in \mathbb{N}$, consider $v^N$ a minimum point for the problem defining $\psi_N(z)$ and define

$$u_n(x^n_i) = v^N(i - Nl)\lambda_n + zNl\lambda_n \qquad \text{if } Nl \leq i \leq N(l+1).$$

We then have

$$\limsup_n E_n(u_n) \leq \psi_N(z) + \frac{c}{N}\sum_{j=1}^K \psi^j(z),$$

and the thesis follows by the arbitrariness of $N$. □

### 1.5 A general convergence theorem

By slightly modifying the proof of Theorem 5 we can easily state a general $\Gamma$-convergence result, allowing a dependence also on $n$ for the energy densities.

**Theorem 6.** *Let $K \geq 1$. Let $\psi^j_n : \mathbb{R} \to [0, +\infty)$ be lower semicontinuous functions and let $p > 1$ exist such that*

$$\psi^1_n(z) \geq c_0(|z|^p - 1), \qquad \psi^j_n(z) \leq c_j(1 + |z|^p). \quad (39)$$

*for all $j \in \{1, \dots, K\}$ and $n \in \mathbb{N}$. For all $N, n \in \mathbb{N}$ let $\psi_{N,n} : \mathbb{R} \to [0, +\infty)$ be defined by*

$$\psi_{N,n}(z) = \min\Big\{\frac{1}{N}\sum_{j=1}^K \sum_{i=0}^{N-j} \psi^j_n\Big(\frac{u(i+j) - u(i)}{j}\Big)$$
$$u : \{0, \dots, N\} \to \mathbb{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K\Big\}. \quad (40)$$

*Suppose that $\psi : \mathbb{R} \to [0, +\infty)$ exists such that*

$$\psi(z) = \lim_N \lim_n \psi^{**}_{N,n}(z) \qquad \textit{for all } z \in \mathbb{R} \tag{41}$$

*(note that this is not restrictive upon passing to a subsequence of $n$ and $N$). Let $E_n$ be defined on $\mathcal{A}_n(0,L)$ by*

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi^j_n \Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big). \tag{42}$$

*Then the $\Gamma$-limit of $E_n$ with respect to the convergence in $L^1(0,L)$ is given by $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \textit{if } u \in W^{1,p}(0,L) \\ +\infty & \textit{otherwise} \end{cases} \tag{43}$$

*on $L^1(0,L)$.*

*Proof.* Let $u_n \to u$ in $L^1(0,L)$. We can repeat the proof for the liminf inequality for Theorem 5, substituting $\psi^j$ by $\psi^j_n$ and $\psi_N$ by $\psi_{N,n}$. We then deduce as in (37)–(38) that

$$\begin{aligned} \liminf_n E_n(u_n) &\geq \Big(1 - \frac{c}{N}\Big) \liminf_n \int_0^L \psi_{N,n}((u^N_n)')\,dt - \frac{c}{N} \\ &\geq \Big(1 - \frac{c}{N}\Big) \int_0^L \psi_N((u^N)')\,dt - \frac{c}{N}, \end{aligned}$$

where $\psi_N = \lim_n \psi^{**}_{N,n}$ and the thesis by letting $N \to +\infty$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such $u$ it suffices to fix $N \in \mathbb{N}$, consider $z_{1,n}, z_{2,n}$ and $\eta_n \in [0,1]$ such that

$$\psi^{**}_{N,n}(z) = \eta_n \psi_{N,n}(z_{1,n}) + (1-\eta_n)\psi_{N,n}(z_{2,n}), \qquad z = \eta_n z_{1,n} + (1-\eta_n) z_{2,n}.$$

Let $v^N_{1,n}$ and $v^N_{2,n}$ be minimum points for the problem defining $\psi_{N,n}(z_{1,n})$ and $\psi_{N,n}(z_{2,n})$, respectively. For the sake of simplicity assume that there exists $m$ such that $mN\eta_n \in \mathbb{N}$ for all $n$. Define

$$u_n(x^n_i) = \begin{cases} v^N_{1,n}(i - Nl)\lambda_n + zmNl\lambda_n & \text{if } mNl \leq i \leq mNl + mN\eta_n \\ v^N_{2,n}(i - Nl - mN\eta_n)\lambda_n + zmNl + z_{1,n} mN\eta_n \lambda_n & \\ & \text{if } mNl + mN\eta_n \leq i \leq mN(l+1). \end{cases}$$

By the growth conditions on $\psi^j_n$ it is easily seen that $(z_{k,n})$ are equibounded and that

$$\sup\{v^N_{k,n}(i) - z_{k,n} i : \ i \in \{0, \dots, N\}, n \in \mathbb{N}\} < +\infty,$$

so that $u_n$ converges to $zx$ uniformly. We then have

$$\limsup_n E_n(u_n) \le L \limsup_n \psi^{**}_{N,n}(z)$$

and the thesis follows by the arbitrariness of $N$. $\Box$

### 1.6 Convergence of minimum problems

We first give a general convergence theorem, and subsequently state a finer theorem for next-to-nearest neighbour interactions.

#### Limit minimum problems on the continuum

From Theorem 6 we immediately deduce the following theorem.

**Theorem 7.** *Let $E_n$ and $F$ be given by Theorem 6, let $f \in L^1(0,L)$ and $d > 0$. Then the minimum values*

$$m_n = \min\Big\{E_n(u) + \int_0^L fu\,dt : \ u(0) = 0,\ u(L) = d\Big\} \tag{44}$$

*converge to*

$$m = \min\Big\{F(u) + \int_0^L fu\,dt : \ u(0) = 0,\ u(L) = d\Big\}, \tag{45}$$

*and from each sequence of minimizers of (44) we can extract a subsequence converging to a minimizer of (45).*

*Proof.* Since the sequence of functionals $(E_n)$ is equi-coercive, it suffices to show that the boundary conditions do not change the form of the $\Gamma$-limit; *i.e.*, that for all $u \in W^{1,p}(0,L)$ such that $u(0) = 0$ and $u(L) = d$ and for all $\varepsilon > 0$ there exists a sequence $u_n$ such that $u_n(0) = 0$, $u_n(L) = d$ and $\limsup_n E_n(u_n) \le F(u) + \varepsilon$.

Let $v_n \to u$ in $L^\infty(0,L)$ be such that $\lim_n E_n(v_n) = F(u)$. With fixed $\eta > 0$ and $N \in \mathbb{N}$ let $K_n \in \mathbb{N}$ be such that

$$\lim_n K_n \lambda_n = \frac{\eta}{N}.$$

For all $l \in \{1, \dots, N\}$ let $\phi_n^{N,l} : [0,L] \to [0,1]$ be the piecewise-affine function defined by $\phi_n^{N,l}(0) = 0$,

$$\phi_n^{N,l\prime} = \begin{cases} 1/(K_n\lambda_n) & \text{on } ((l-1)K_n\lambda_n, lK_n\lambda_n) \\ -1/(K_n\lambda_n) & \text{on } ((n-lK_n)\lambda_n, (n-lK_n+K_n)\lambda_n) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$u_n^{N,l} = \phi_n^{N,l} v_n + (1 - \phi_n^{N,l}) u.$$

We have

$$\begin{aligned}
E_n(u_n^{N,l}) \le E_n(u_n) &+ c\Big(\int_0^{\eta+K\lambda_n} (1+|u'|^p)\,dt + \int_{L-\eta-K\lambda_n}^{L} (1+|u'|^p)\,dt\Big) \\
&+ c\Big(\int_{((l-1)K_n-K)\lambda_n,(lK_n+K)\lambda_n)\cap(0,L)} |u_n'|^p\,dt \\
&+ \int_{((n-lK_n-K)\lambda_n,(n-lK_n+K_n+K)\lambda_n)\cap(0,L)} |v_n'|^p\,dt \\
&+ \int_0^L \frac{1}{(K_n\lambda_n)^p}|v_n-u|^p\Big) \\
\le E_n(u_n) &+ c\Big(\int_0^{2\eta} (1+|u'|^p)\,dt + \int_{L-2\eta}^{L} (1+|u'|^p)\,dt\Big) \\
&+ c\Big(\int_{(((l-2)\eta/N,((l+1)\eta/N)\cup(L-(l+1)\eta/N,L-(l-2)\eta/N))\cap(0,L)} |v_n'|^p\,dt\Big) \\
&+ c\frac{N^p}{\eta^p}\|v_n-u\|^p_{L^\infty(0,L)}
\end{aligned}$$

for $n$ large enough. Since

$$\begin{aligned}
&\sum_{l=1}^{N} \int_{((l-2)\eta/N,((l+1)\eta/N)\cup(L-(l+1)\eta/N,L-(l-2)\eta/N)\cap(0,L)} |u_n'|^p\,dt \\
&\le 2\int_0^L (1+|v_n'|^p)\,dt \le c,
\end{aligned}$$

for all $n$ there exists $l_n \in \{1,\dots,N\}$ such that

$$\begin{aligned}
E_n(u_n^{N,l_n}) \le E_n(v_n) &+ c\Big(\int_0^{2\eta}(1+|u'|^p)\,dt + \int_{L-2\eta}^{L}(1+|u'|^p)\,dt\Big) \\
&+ \frac{c}{N} + c\frac{N^p}{\eta^p}\|v_n-u\|^p_{L^\infty(0,L)}
\end{aligned}$$

Setting $u_n = u_n^{N,l_n}$ we then have

$$\limsup_n E_n(u_n) \le F(u) + c\Big(\int_0^{2\eta}(1+|u'|^p)\,dt + \int_{L-2\eta}^{L}(1+|u'|^p)\,dt\Big) + \frac{c}{N},$$

and the desired inequality by the arbitrariness of $\eta$ and $N$. □

### Next-to-nearest interactions: phase transitions and boundary layers

If the function $\psi$ giving the limit energy density in Theorem 6 is not strictly convex, converging sequences of minimizers of problems of the type (44) may converge to particular minimizers of (45). This happens in the case of next-to-nearest interactions, where the formula giving $\psi$ is of particular help.

We examine the case when $\tilde{\psi}$ in (22) is not convex and of minimum problems (44) with $f = 0$. Upon some change of coordinates it is not restrictive to examine problems of the form

$$m_n = \min\{E_n(u):\ u(0) = 0,\ u(L) = 0\}, \tag{46}$$

and to suppose

(H1) we have

$$\min \tilde{\psi} = \tilde{\psi}(1) = \tilde{\psi}(-1). \tag{47}$$

For the sake of simplicity we make the additional assumptions

(H2) we have

$$\tilde{\psi}(z) > 0 \text{ if } |z| \neq 1; \tag{48}$$

(H3) there exist unique $z_1^+, z_2^+$ and $z_1^-, z_2^-$ such that

$$\psi^2(\pm 1) + \frac{1}{2}\Big(\psi^1(z_1^\pm) + \psi^1(z_2^\pm)\Big) = \min \tilde{\psi}, \qquad z_1^\pm, z_2^\pm = \pm 2;$$

We set

$$\mathbf{M}^+ = \{(z_1^+, z_2^+), (z_2^+, z_1^+)\}, \qquad \mathbf{M}^- = \{(z_1^-, z_2^-), (z_2^-, z_1^-)\} \tag{49}$$

$$\mathbf{M} \ = \ \mathbf{M}^+ \cup \mathbf{M}^-. \tag{50}$$

(H4) we have $z_i^+ \neq z_j^-$ for all $i, j \in \{1, 2\}$;

(H5) all functions are $C^1$.

Under hypotheses (H1)–(H2) Theorem 6 simply gives that $m_n \to 0$ and that the limits $u$ of minimizers satisfy $|u'| \leq 1$ a.e. We will see that indeed they are 'extremal' solutions to the problem

$$\min\{F(u):\ u(0) = 0,\ u(L) = 0\}. \tag{51}$$

The effect of the non validity of hypotheses (H3)–(H5) is explained in Remark 6.

The key idea is that it is energetically convenient for discrete minimizer to remain close to the two states minimizing $\tilde{\psi}$, and that every time we have a transition from one of the two minimal configurations to the other a fixed amount of energy is spent (independent of $n$). To exactly quantify this fact we introduce some functions and quantities.

**Definition 2 (Minimal energy configurations).** *Let* $\mathbf{z} = (z_1, z_2) \in \mathbf{M}$*; we define* $u^{\mathbf{z}} : \mathbb{Z} \to \mathbb{R}$ *by*

$$u^{\mathbf{z}}(i) = \Big[\frac{i}{2}\Big] z_2 + \Big(i - \Big[\frac{i}{2}\Big]\Big) z_1, \tag{52}$$

*and* $u_n^{\mathbf{z}} : \lambda_n \mathbb{Z} \to \mathbb{R}$ *by*

$$u_n^{\mathbf{z}}(x_i^n) = u^{\mathbf{z}}(i)\lambda_n \tag{53}$$

**Definition 3 (Crease and boundary-layer energies).** Let $v : \mathbb{Z} \to \mathbb{R}$. The *right-hand side boundary layer energy of* $v$ is

$$B_+(v) = \inf_{N\in\mathbb{N}} \min\Big\{\sum_{i\geq 0}\Big(\psi^2\Big(\frac{u(i+2)-u(i)}{2}\Big) + \psi^1(u(i+1)-u(i)) - \min\tilde\psi\Big) \\ : u : \mathbb{N}\cup\{0\} \to \mathbb{R},\ u(i) = v(i) \text{ if } i \geq N\Big\},$$

The *left-hand side boundary layer energy of* $v$ is

$$B_-(v) = \inf_{N\in\mathbb{N}} \min\Big\{\sum_{i\leq 0}\Big(\psi^2\Big(\frac{u(i)-u(i-2)}{2}\Big) + \psi^1(u(i)-u(i-1)) - \min\tilde\psi\Big) \\ : u : -\mathbb{N}\cup\{0\} \to \mathbb{R},\ u(i) = v(i) \text{ if } i \leq -N\Big\},$$

Let $v^\pm : \mathbb{Z} \to \mathbb{R}$. The *transition energy between* $v^-$ *and* $v^+$ is

$$C(v^-, v^+) = \inf_{N\in\mathbb{N}} \min\Big\{\sum_{i\in\mathbb{Z}}\Big(\psi^2\Big(\frac{u(i+2)-u(i)}{2}\Big) + \psi^1(u(i+1)-u(i)) - \min\tilde\psi\Big) \\ : u : \mathbb{Z} \to \mathbb{R},\ c^\pm \in \mathbb{R}, u(i) = v^\pm(i) + c^\pm \text{ if } \pm i \geq N\Big\}.$$

*Remark 5.* Condition (H4) implies that $C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) > 0$ and $C(u^{\mathbf{z}^-}, u^{\mathbf{z}^+}) > 0$ if $\mathbf{z}^\pm \in \mathbf{M}^\pm$.

We can now describe the behaviour of minimizing sequences for (44). A more general statement can be found in [16].

**Theorem 8.** *Suppose that* (H1)–(H5) *hold. We then have:*

(Case $n$ even) *The minimizers* $(u_n)$ *of* (44) *for* $n$ *even converge, up to subsequences, to one of the functions*

$$\overline{u}_+(x) = \begin{cases} x & \text{if } 0 \leq x \leq L/2 \\ L - x & \text{if } L/2 \leq x \leq L, \end{cases} \qquad \overline{u}_-(x) = \begin{cases} -x & \text{if } 0 \leq x \leq L/2 \\ -(L - x) & \text{if } L/2 \leq x \leq L. \end{cases}$$

*Let*

$$D := \min\Big\{B_+(u^{\mathbf{z}^+}) + C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) + B_-(u^{\mathbf{z}^-}), \\ B_+(u^{\mathbf{z}^-}) + C(u^{\mathbf{z}^-}, u^{\mathbf{z}^+}) + B_-(u^{\mathbf{z}^+}) :\ \mathbf{z}^+ \in \mathbf{M}^+,\ \mathbf{z}^- \in \mathbf{M}^-\}.$$

*If* $(u_n)$ *converges (up to subsequences) to* $\overline{u}_\pm$ *then there exist* $\mathbf{z}^+ \in \mathbf{M}^+$, *and* $\mathbf{z}^- \in \mathbf{M}^-$ *such that*

$$D = B_+(u^{\mathbf{z}^+}) + C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) + B_-(u^{\mathbf{z}^-}) \tag{54}$$

*and*

$$E_n(u_n) = D\,\lambda_n + o(\lambda_n). \tag{55}$$

(Case $n$ odd) *In the case $n$ odd the same conclusions hold, upon substituting terms of the form*

$$B_+(u^{\mathbf{z}^\pm}) + C(u^{\mathbf{z}^\pm}, u^{\mathbf{z}^\mp}) + B_-(u^{\mathbf{z}^\mp})$$

*by terms of the form*

$$B_+(u^{\mathbf{z}^\pm}) + C(u^{\mathbf{z}^\pm}, u^{\mathbf{z}^\mp}) + B_-(u^{\overline{\mathbf{z}^\mp}}),$$

*where we have set* $\overline{(z_1, z_2)} = (z_2, z_1)$.

*Proof.* We only deal with the case $n$ even, as the case $n$ odd is dealt with similarly.

Let $u_n$ be a minimizer for (44). We may assume that $u_n$ converge in $W^{1,p}(0, L)$ and uniformly. By comparison with $E_n(\overline{u})$ we have

$$E_n(u_n) \le L \min \tilde{\psi} + c\lambda_n. \tag{56}$$

We can consider the scaled energies

$$E^1_n(u) = \frac{1}{\lambda_n}(E_n(u) - L \min \tilde{\psi}). \tag{57}$$

Note that we have

$$\begin{aligned} E^1_n(u) = \sum_{i=0}^{n-2} \Big( & \psi^2\Big(\frac{u_{i+2} - u_i}{2\lambda_n}\Big) \\ & + \frac{1}{2}\Big(\psi^1\Big(\frac{u_{i+2} - u_{i+1}}{\lambda_n}\Big) + \psi^1\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big)\Big) - \min \tilde{\psi}\Big) \\ & + \frac{1}{2}\Big(\psi^1\Big(\frac{u_n - u_{n-1}}{\lambda_n}\Big) + \psi^1\Big(\frac{u_1 - u_0}{\lambda_n}\Big)\Big) - \min \tilde{\psi}. \end{aligned} \tag{58}$$

From (56) and (58) we deduce that

$$\begin{aligned} & \sum_{i=0}^{n-2} \Big(\psi^2\Big(\frac{u_n(x^n_{i+2}) - u_n(x^n_i)}{2\lambda_n}\Big) \\ & + \frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2}) - u_n(x^n_{i+1})}{\lambda_n}\Big) + \psi^1\Big(\frac{u_n(x^n_{i+1}) - u_n(x^n_i)}{\lambda_n}\Big)\Big) - \min \tilde{\psi}\Big) \le c. \end{aligned}$$

We infer that for every $\eta > 0$ we have that if we denote by $I_n(\eta)$ the set of indices $i$ such that

$$\begin{aligned} & \psi^2\Big(\frac{u_n(x^n_{i+2}) - u_n(x^n_i)}{2\lambda_n}\Big) \\ & + \frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2}) - u_n(x^n_{i+1})}{\lambda_n}\Big) + \psi^1\Big(\frac{u_n(x^n_{i+1}) - u_n(x^n_i)}{\lambda_n}\Big)\Big) \le \min \tilde{\psi} + \eta \end{aligned}$$

then

$$\sup_n I_n(\eta) < +\infty.$$

Let $\varepsilon = \varepsilon(\eta)$ be defined so that if

$$\psi^2\Big(\frac{z_1+z_2}{2}\Big)+\frac{1}{2}\Big(\psi^1(z_1)+\psi^1(z_2)\Big) - \min\tilde\psi \le \eta$$

then

$$\mathrm{dist}\,((z_1,z_2),\mathbf{M}) \le \varepsilon(\eta).$$

Choose $\eta > 0$ so that

$$2\varepsilon(\eta) < \min\{|\mathbf{z}^+ - \mathbf{z}^-|,\ \mathbf{z}^+ \in \mathbf{M}^+,\ \mathbf{z}^- \in \mathbf{M}^-\}.$$

We then deduce that if $i-1, i \notin I_n(\eta)$ then there exists $\mathbf{z} \in \mathbf{M}$ such that

$$\Big|\Big(\frac{u_n(x^n_{i+1}) - u_n(x^n_i)}{\lambda_n}, \frac{u_n(x^n_{i+2}) - u_n(x^n_{i+1})}{\lambda_n}\Big) - \mathbf{z}\Big| \le \varepsilon$$

and

$$\Big|\Big(\frac{u_n(x^n_{i}) - u_n(x^n_{i-1})}{\lambda_n}, \frac{u_n(x^n_{i+1}) - u_n(x^n_{i})}{\lambda_n}\Big) - \overline{\mathbf{z}}\Big| \le \varepsilon$$

Hence, there exist a finite number of indices $0 = i_0 < i_1 < i_2 < \cdots < i_{N_n} = n$ such that for all $j = 1, \ldots, N_n$ there exists $\mathbf{z}^n_j \in \mathbf{M}$ such that for all $i \in \{i_{j-1}+1, \ldots, i_j - 1\}$ we have

$$\Big|\Big(\frac{u_n(x^n_{i+1}) - u_n(x^n_i)}{\lambda_n}, \frac{u_n(x^n_{i+2}) - u_n(x^n_{i+1})}{\lambda_n}\Big) - \mathbf{z}^n_j\Big| \le \varepsilon.$$

Let $\{j_0, j_1, \ldots, j_{M_n}\}$ be the maximal subset of $\{i_0, i_1, \ldots, i_{N_n}\}$ defined by the requirement that if $z^n_{j_k} \in \mathbf{M}^{\pm}$ then $z^n_{j_k+1} \in \mathbf{M}^{\mp}$. Note that in this case we deduce that $E_n(u_n) \ge cM_n$, so that $M_n$ are equi-bounded. Upon choosing a subsequence we may then suppose $M_n = M$ independent of $n$, and also that $x^n_{j_k} \to x_k \in [0, L]$ and $\mathbf{z}^n_{j_k} = \mathbf{z}_k$. By the arbitrariness of $\eta$ we deduce that $\lim_n u_n = u$, and $u$ is characterized by $u(0) = u(L) = L$ and $u' = \pm 1$ on $(x_{k-1}, x_k)$, the sign determined by whether $\mathbf{z}_k \in \mathbf{M}^+$ or $\mathbf{z}_k \in \mathbf{M}^+$. Let $y_0 = 0 < y_1 < \cdots < y_N = L$ be distinct ordered points such that $\{y_i\} = \{x_k\}$ (the set of indices may be different if $x_k = x_{k+1}$ for some $k$). Choose indices $k_1, \ldots, k_N$ such that $x^n_{k_j} \to (y_{j-1} + y_j)/2$. Let $\mathbf{z}_j$ be the limit of $\mathbf{z}^n_{j_k}$ related to the interval $(y_j, y_{j+1})$. We then have, for a suitable continuous $\omega : [0, +\infty) \to [0, +\infty)$,

$$\begin{aligned}&\sum_{i=0}^{k_1-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2}) - u_n(x^n_i)}{2\lambda_n}\Big)\\ &+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2}) - u_n(x^n_{i+1})}{\lambda_n}\Big) + \psi^1\Big(\frac{u_n(x^n_{i+1}) - u_n(x^n_i)}{\lambda_n}\Big)\Big) - \min\tilde\psi\Big)\\ &\ge B_+(u^{\mathbf{z}_1}) - \omega(\varepsilon),\end{aligned}$$

$$\sum_{i=k_j}^{k_{j+1}-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_i)}{2\lambda_n}\Big)$$
$$+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_i)}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)$$
$$\geq C(u^{\mathbf{z}_j},u^{\mathbf{z}_{j+1}})-\omega(\varepsilon)\text{ for all }j\in\{1,\dots,N-1\},$$

$$\sum_{i=k_N}^{n-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_i)}{2\lambda_n}\Big)$$
$$+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_i)}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)$$
$$\geq B_-(u^{\mathbf{z}_N})-\omega(\varepsilon).$$

By the arbitrariness of $\varepsilon$ and the definition of $D$ we easily get $\liminf_n E^1_n(u_n)\geq D$, and by Remark 5 that if $u\neq\overline{u}_\pm$ then $\liminf_n E^1_n(u_n)>D$.

It remains to show that $\limsup_n E^1_n(u_n)\leq D$; *i.e.*, for every fixed $\eta>0$ to exhibit a sequence $\overline{u}_n$ such that $\overline{u}_n(0)=\overline{u}_n(L)=0$ and $\limsup_n E^1_n(\overline{u}_n)\leq D+c\eta$. Suppose that

$$D=B_+(u^{\mathbf{z}^+})+C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})+B_-(u^{\mathbf{z}^-}),$$

with $\mathbf{z}^+=(z_1^+,z_2^+)$, $\mathbf{z}^-=(z_1^-,z_2^-)$, the other cases being dealt with in the same way. Let $\eta>0$ be fixed and let $N\in\mathbb{N}$, $v_+,v_-,v:\mathbb{Z}\to\mathbb{R}$ be such that

$$v_+(i)=u^{\mathbf{z}^+}(i)\qquad\text{for }i\geq N,\qquad v_-(i)=u^{\mathbf{z}^-}(i)\qquad\text{for }i\leq -N,$$

$$v(i)=\begin{cases}u^{\mathbf{z}^+}(i) & \text{for }i\leq -N\\ u^{\mathbf{z}^-}(i) & \text{for }i\geq N,\end{cases}$$

and

$$\sum_{i\geq 0}\Big(\psi^2\Big(\frac{v_+(i+2)-v_+(i)}{2}\Big)+\psi^1(u(i+1)-u(i))-\min\tilde\psi\Big)\leq B_+(u^{\mathbf{z}^+})+\eta$$

$$\sum_{i\leq 0}\Big(\psi^2\Big(\frac{v_-(i)-v_-(i-2)}{2}\Big)+\psi^1(u(i)-u(i-1))-\min\tilde\psi\Big)\leq B_-(u^{\mathbf{z}^-})+\eta$$

$$\sum_{i\in\mathbb{Z}}\Big(\psi^2\Big(\frac{v(i+2)-v(i)}{2}\Big)+\psi^1(v(i+1)-v(i))-\min\tilde\psi\Big)\leq C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})+\eta.$$

We then set

$$\overline{u}(x_i^n) = \begin{cases} (v_+(i) - v_+(0))\lambda_n & \text{if } i \le N \\ u_n^{\mathbf{z}^+}(x_i^n) - v_+(0)\lambda_n + z_n^1(x_i^n - x_N^n) & \text{if } N \le i \le \frac{n}{2} - N \\ v\Big(i - \frac{n}{2}\Big)\lambda_n - \frac{L}{2} & \text{if } \frac{n}{2} - N \le i \le \frac{n}{2} + N \\ u_n^{\mathbf{z}^-}(x_{n-i}^n) - v_-(0)\lambda_n + z_n^2(x_i^n - x_{n-N}^n) & \text{if } \frac{n}{2} + N \le i \le n - N \\ (v_-(n-i) - v_-(0))\lambda_n & \text{if } n - N \le i \le n, \end{cases}$$

where

$$z_n^1 = \frac{u^{\mathbf{z}^+}\left(\frac{n}{2}\right)\lambda_n - \frac{L}{2} + v_+(0)\lambda_n}{\left(\frac{n}{2} - 2N\right)\lambda_n}$$

$$z_n^2 = \frac{u^{\mathbf{z}^-}\left(\frac{n}{2}\right)\lambda_n + \frac{L}{2} + v_-(0)\lambda_n}{\left(\frac{n}{2} - 2N\right)\lambda_n}.$$

Note that $\lim_n z_n^1 = \lim_n z_n^2 = 0$. Using (H5) we easily get the desired inequality. □

*Remark 6.* From the proof above it can be easily seen that hypotheses (H3)–(H5) may be relaxed at the expense of a heavier notation and some changes in the results. Clearly, if (H3) does not hold then the sets of minimal pairs $\mathbf{M}^+$, $\mathbf{M}^-$ are larger, and the definition of $D$ must be changed accordingly, possibly taking into account also more than one transition.

If hypothesis (H4) does not hold then $C(u^{\mathbf{z}^+}, u^{\mathbf{z}^-}) = C(u^{\mathbf{z}^-}, u^{\mathbf{z}^+}) = 0$ for some $\mathbf{z}^+ \in \mathbf{M}^+$, $\mathbf{z}^- \in \mathbf{M}^-$. In this case the energetic analysis of $E_n^1$ is not sufficient to characterize the minimizers, as we have no control on the number of transitions between $u' = 1$ and $u' = -1$.

Condition (H5) has been used to construct the recovery sequence $(\overline{u}_n)$. It can be relaxed to assuming that $\tilde{\psi}$ is smooth at $\pm 1$; more precisely, it suffices to suppose that

$$\lim_{z \to \pm 1} \frac{\tilde{\psi}(z) - \min \tilde{\psi}}{|z \mp 1|} = 0. \tag{59}$$

If this condition does not hold the value $D$ is given by a more complex formula, where we take into account also the values at 0 of the solutions of the boundary layer terms.

The proof of Theorem 8 suggests the corresponding $\Gamma$-limit result for $E_n^1$. The details in a much more general setting can be found in [16].

### 1.7 More examples

In this section we examine some situations when some of the hypotheses considered hitherto are relaxed. Namely,

(i) (*weak nearest-neighbour interactions*) when the condition

$$\psi_n^1(z) \ge c_0(|z|^p - 1)$$

does not hold. In this case, the limit energy may be defined on a set of vector functions (multi-phase limit);

(ii) (*very-long-range interactions*) when the energy $E_n$ takes into account interactions up to the order $K_n$ with $K_n \to +\infty$. In this case, the limit energy may be non-local;

(iii) (*non spatially homogeneous interactions*) when the interaction between $u_i$ and $u_{i+j}$ may depend also on $i$. In this case a homogenization process may take place.

For the sake of presentation we will explicitly treat only the case of quadratic energies, of the form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \rho_n^{j,i} \Big( \frac{u_{i+j} - u_i}{j\lambda_n} \Big)^2, \tag{60}$$

with $\rho_n^{j,i} > 0$ and $1 \leq K_n \leq n$.

*Remark 7.* In the case when $\rho_n^{j,i} = \rho^j$, $K_n = K$ and $\rho^1 > 0$ then the $\Gamma$-limit in Theorem 2 of $E_n$ is given by

$$F(u) = \rho \int_0^L |u'|^2 \, dt, \qquad \text{where } \rho = \sum_{j=1}^{K} \rho^j.$$

The same conclusion holds if $\rho_n^{j,i} = \rho^j$, $K_n = n$, $\rho^1 > 0$, and $\rho = \sum_{j=1}^{\infty} \rho^j$.

**Weak nearest-neighbour interactions: multi-phase limits**

We only treat the case of next-to-nearest neighbour interactions with weak nearest-neighbour interactions; *i.e.*, in (60) we take $K_n = 2$, $\rho_n^{2,i} = c_2$, and $\rho_n^{1,i} = a_n$ with

$$\lim_n \frac{a_n}{\lambda_n^2} = c_1.$$

The energies we consider take the form

$$E_n(u) = c_2 \sum_{i=0}^{n-2} \lambda_n \Big( \frac{u_{i+2} - u_i}{2\lambda_n} \Big)^2 + \sum_{i=0}^{n-1} \lambda_n a_n \Big( \frac{u_{i+1} - u_i}{\lambda_n} \Big)^2. \tag{61}$$

For all $n$ $u_n \in \mathcal{A}_n(0, L)$, we consider $u_{n,\mathrm{e}}, u_{n,\mathrm{o}} : \{0, \dots, [n/2]\} \to \mathbb{R}$, defined by

$$u_{n,\mathrm{e}}(i) = u_n(2i\lambda_n), \qquad u_{n,\mathrm{o}}(i) = u_n((2i+1)\lambda_n)$$

(for simplicity, $u_n(x_i^n) = u_n(L)$ if $i > n$), which take into account the values of $u_n$ on even and odd points, respectively. Note that the energy $E_n(u_n)$ can be identified with an energy $E_n(u_{n,\mathrm{e}}, u_{n,\mathrm{o}})$ defined by

$$
\begin{aligned}
E_n(u_{n,\mathrm{e}},u_{n,\mathrm{o}}) = c_2 &\sum_{i=0}^{[n/2]-1} \lambda_n\Big(\frac{u_{n,\mathrm{e}}(i+1)-u_{n,\mathrm{e}}(i)}{2\lambda_n}\Big)^2\\
&+c_2 \sum_{i=0}^{[n/2]-1} \lambda_n\Big(\frac{u_{n,\mathrm{o}}(i+1)-u_{n,\mathrm{o}}(i)}{2\lambda_n}\Big)^2\\
&+ \sum_{i=0}^{[n/2]-1} \lambda_n a_n\Big(\frac{u_{n,\mathrm{o}}(i)-u_{n,\mathrm{e}}(i)}{\lambda_n}\Big)^2\\
&+ \sum_{i=0}^{[n/2]-1} \lambda_n a_n\Big(\frac{u_{n,\mathrm{o}}(i)-u_{n,\mathrm{e}}(i+1)}{\lambda_n}\Big)^2. \qquad (62)
\end{aligned}
$$

We say that the sequence $(u_n)$ converges (in $L^1(0,L)$) to $u$ to the pair $(u_{\mathrm{e}},u_{\mathrm{o}})$ if the piecewise-affine interpolates $\tilde u_{n,\mathrm{e}},\tilde u_{n,\mathrm{o}}$ defined by

$$
\begin{aligned}
\tilde u'_{n,\mathrm{e}} &= \frac{u_{n,\mathrm{e}}(i+1)-u_{n,\mathrm{e}}(i)}{2\lambda_n} \qquad \text{on } (x_n^{2i},x_n^{2i+2}),\\
\tilde u'_{n,\mathrm{o}} &= \frac{u_{n,\mathrm{o}}(i+1)-u_{n,\mathrm{o}}(i)}{2\lambda_n} \qquad \text{on } (x_n^{2i},x_n^{2i+2}),
\end{aligned}
$$

respectively, converge to $(u_{\mathrm{e}},u_{\mathrm{o}})$, respectively. We then have the following result.

**Theorem 9.** *The energies $E_n$ $\Gamma$-converge with respect to the convergence of $u_n$ to $(u_{\mathrm{e}},u_{\mathrm{o}})$, to the functional*

$$
F(u_{\mathrm{e}},u_{\mathrm{o}}) = \begin{cases} \displaystyle\frac12 c_2\int_0^L |u'_{\mathrm{e}}|^2\,dt + \frac12 c_2\int_0^L |u'_{\mathrm{o}}|^2\,dt + c_1\int_0^L |u_{\mathrm{e}}-u_{\mathrm{o}}|^2\,dt \\ \hfill \textit{if } u_{\mathrm{e}},u_{\mathrm{o}}\in H^1(0,L)\\ +\infty \hfill \textit{otherwise.}\end{cases}
$$

*If $c_1=+\infty$ the formula above is understood to mean that $F(u_{\mathrm{e}},u_{\mathrm{o}})=+\infty$ if $u_{\mathrm{e}}\neq u_{\mathrm{o}}$, so that, having set $u=u_{\mathrm{e}}=u_{\mathrm{o}}$ we recover for $F$ the form*

$$
F(u) = \begin{cases} \displaystyle c_2\int_0^L |u'|^2\,dt & \textit{if } u\in H^1(0,L)\\ +\infty & \textit{otherwise.}\end{cases}
$$

*Proof.* It suffices to treat the case $a_n = c_1\lambda_n^2$ with $c_1<+\infty$, as all the others are easily obtained from that by a comparison argument. To obtain the liminf inequality, it suffices to use Theorem 1 for the first two terms in (62) and note that each of the last two terms converges to

$$
\frac12 c_1\int_0^L |u_{\mathrm{e}}-u_{\mathrm{o}}|^2\,dt,
$$

as the convergence of $\tilde u_{n,\mathrm{e}},\tilde u_{n,\mathrm{o}}$ to $(u_{\mathrm{e}},u_{\mathrm{o}})$, respectively, is uniform.

The limsup inequality is obtained by direct computations on piecewise-affine functions, and then reasoning by density as usual. □

**Very-long interactions: non-local limits**

For all $n \in \mathbb{N}$ let $\rho_n : \lambda_n\mathbb{Z} \to [0, +\infty)$. We consider the following form of the discrete energies

$$E_n(u) = \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L]\\ x\neq y}} \lambda_n\rho_n(x-y)\Big(\frac{u(x)-u(y)}{x-y}\Big)^2 \tag{63}$$

defined for $u : \lambda_n\mathbb{Z} \to \mathbb{R}$. Note that we may assume that $\rho_n$ is an even function, upon replacing $\rho_n(z)$ by $\tilde\rho_n(z) = (1/2)(\rho_n(z) + \rho_n(-z))$. We will tacitly make this simplifying assumption in the sequel.

We will consider the following hypotheses on $\rho_n$:

(H1) (*equi-coerciveness of nearest-neighbour interactions*) $\inf_n \rho_n(\lambda_n) > 0$;

(H2) (*local uniform summability of* $\rho_n$) for all $T > 0$ we have

$$\sup_n \sum_{x\in\lambda_n\mathbb{Z}\cap(0,T)} \rho_n(x) < +\infty.$$

*Remark 8.* Note that (H2) can be rephrased as a local uniform integrability property for $\lambda_n\rho_n$ on $\mathbb{R}^2$: for all $T > 0$

$$\sup_n \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\\ x\neq y, |x|,|y|\leq T}} \lambda_n\rho_n(x-y) < +\infty.$$

As a consequence, if (H2) holds then, up to a subsequence, we can assume that the Radon measures

$$\mu_n = \sum_{x,y\in\lambda_n\mathbb{Z},\ x\neq y} \lambda_n\rho_n(x-y)\delta_{(x,y)}$$

($\delta_z$ denotes the Dirac mass at $z$) locally converge weakly in $\mathbb{R}^2$ to a Radon measure $\mu_0$, and that the Radon measures

$$\beta_n = \sum_{z\in\lambda_n\mathbb{Z}} \rho_n(z)\delta_z$$

locally converge weakly in $\mathbb{R}$ to a Radon measure $\beta_0$. These two limit measures are linked by the relation

$$\mu_0(A) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} |A_s|\,d\beta_0(s), \tag{64}$$

where $|A_s|$ is the Lebesgue measure of the set

$$A_s = \{t \in \mathbb{R} : (s(e_1 - e_2) + t(e_1 + e_2))/\sqrt{2} \in A\}.$$

If (H1) holds then we have the orthogonal decomposition

$$\beta_0 = \beta_1 + c_1\delta_0, \tag{65}$$

for some $c_1 > 0$ and a Radon measure $\beta_1$ on $\mathbb{R}$. We also denote

$$\mu = \mu_0 \llcorner (\mathbb{R}^2 \setminus \Delta) \tag{66}$$

(the restriction of $\mu_0$ to $\mathbb{R}^2 \setminus \Delta$), where $\Delta = \{(x,x) : x \in \mathbb{R}\}$. By the decomposition above, we have

$$\mu_0 = \mu + \frac{1}{\sqrt{2}} c_1 \mathcal{H}^1 \llcorner \Delta,$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

The main result of this section is the following.

**Theorem 10 (Compactness and representation).** *If conditions* (H1) *and* (H2) *hold, then there exist a subsequence (not relabeled), a Radon measure $\mu$ on $\mathbb{R}^2$ and a constant $c_1 > 0$ such that the energies $E_n$ $\Gamma$-converge to the energy $F$ defined on* $\mathrm{L}^1(0,L)$ *by*

$$F(u) = \begin{cases} c_1 \displaystyle\int_{(0,L)} |u'|^2\,dt + \int_{(0,L)^2} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y) & \text{if } u \in W^{1,2}(0,L) \\ +\infty & \text{otherwise,} \end{cases} \tag{67}$$

*wit respect to convergence in measure and $L^1(0,L)$, where the measure $\mu$ and $c_1$ are given by* (66) *and* (65)*, respectively.*

*Proof.* Upon passing to a subsequence we may assume that the measures $\mu_n$ in Remark 8 converge to $\mu_0$. Then, $\mu$ and $c_1$ given by (66) and (65) are well defined. Hence, it suffices to prove the representation for the $\Gamma$-limit along this sequence.

We begin by proving the liminf inequality. Let $u_n \to u$ in $\mathrm{L}^1(0,L)$ be such that $\sup_n E_n(u_n) < +\infty$. By hypothesis (H1), the sequence $u_n$ converges weakly in $W^{1,2}((0,L)$.

With fixed $m \in \mathbb{N}$, we have the equality

$$\begin{aligned} E_n(u_n) \;=\; & \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L] \\ |x-y|\le 1/m,\, x\neq y}} \rho_n(x-y)\lambda_n \Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\ & + \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L] \\ |x-y|> 1/m}} \rho_n(x-y)\lambda_n \Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\ =: & \; I_n^1(u_n) + I_n^2(u_n). \end{aligned} \tag{68}$$

We now estimate these two terms separately.

We first note that there exist positive $\alpha_n$ converging to 0 such that

$$\lim_n 2 \sum_{k=1}^{[\alpha_n/\lambda_n]} \rho_n(\lambda_n k) \geq c_1 - \frac{1}{m}.$$

Let $(a,b) \subset (0,L)$. For all $N \in \mathbb{N}$ and for $n$ large enough we then have

$$\begin{aligned}
I_n^1(u_n) &\geq \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap(a,b)\\ |x-y|\leq\alpha_n,\ x\neq y}} \lambda_n\rho_n(x-y)\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\
&\geq \sum_{i=1}^{N} 2 \sum_{k=1}^{[\alpha_n/\lambda_n]} \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap(y_{i-1},y_i)\\ |x-y|=\lambda_n k}} \lambda_n\rho_n(\lambda_n k)\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\
&\geq \sum_{i=1}^{N} 2 \sum_{k=1}^{[\alpha_n/\lambda_n]} \frac{(b-a)}{N}\rho_n(\lambda_n k)\Big(\frac{u(y_i)-u(y_{i-1})}{y_i-y_{i-1}}\Big)^2 + o(1)
\end{aligned}$$

as $n \to \infty$, where we have set

$$y_i = a + \frac{i}{N}(b-a),$$

we have used the fact that $u_n \to u$ uniformly and the convexity of $z \mapsto z^2$. This shows that

$$\liminf_n I_n^1(u_n) \geq \Big(c_1 - \frac{1}{m}\Big)\int_{(a,b)} |u'|^2\,dt.$$

From this inequality we obtain that

$$\liminf_n I_n^1(u_n) \geq \Big(c_1 - \frac{1}{m}\Big)\int_{(0,L)} |u'|^2\,dt.$$

As for the second term, for all $\eta > 0$ let $\Delta_\eta = \{(x,y) \in \mathbb{R}^2 :\ |x-y| > \eta\}$. Note that the convergence

$$\frac{u_n(x)-u_n(y)}{x-y} \quad\longrightarrow\quad \frac{u(x)-u(y)}{x-y}$$

is uniform on $(0,L)^2 \setminus \Delta_\eta$, so that, by the weak convergence of $\mu_n$ we have

$$\liminf_n I_n^2(u_n) \geq \int_{(0,L)^2\setminus\Delta_{1/m}} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

By summing up all these inequalities and letting $m \to +\infty$ we eventually get

$$\liminf_n E_n(u_n) \ge c_1 \int_{(0,L)} |u'|^2\,dt + \int_{(0,L)^2} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

To prove the limsup inequality it suffices to show it for piecewise-affine functions, since this set is strongly dense in the space of piecewise $W^{1,2}$ functions. In this case it suffices to take $u_n = u$. □

### Homogenization

We only treat the case of nearest-neighbour interactions; *i.e.*, in (60) we take $K_n = 1$ and $\rho_n^{1,i} = \rho_i$ with $i \mapsto \rho_i$ defining a $M$-periodic function $\mathbb{Z} \to \mathbb{R}$:

$$\rho_{i+M} = \rho_i.$$

The energies we consider take the form

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \rho_i \Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)^2. \tag{69}$$

**Theorem 11.** *The energies $E_n$ $\Gamma$-converge to the energy defined by*

$$F(u) = \begin{cases} \overline{\rho} \displaystyle\int_0^L |u'|^2\,dt & \text{if } u \in H^1(0,L) \\ +\infty & \text{otherwise,} \end{cases}$$

*where*

$$\overline{\rho} = M\Big(\sum_{i=1}^M \frac{1}{\rho_i}\Big)^{-1}.$$

*Proof.* Note that

$$\overline{\rho} = \min\Big\{ M\sum_{i=1}^M \rho_i z_i^2 :\ \sum_{i=1}^M z_i = 1\Big\},$$

so that

$$\overline{\rho} z^2 = \min\Big\{ M\sum_{i=1}^M \rho_i z_i^2 :\ \sum_{i=1}^M z_i = z\Big\}.$$

We then immediately have

$$\sum_{i=0}^{n-1} \lambda_n \rho_i \Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)^2 \ge \sum_{i=0}^{[n/M]-1} M\lambda_n \overline{\rho} \Big(\frac{u_{M(i+1)}-u_{Mi}}{M\lambda_n}\Big)^2,$$

which gives the liminf inequality.

The limsup inequality for the function $u(x) = zx$ is obtained by choosing $u_n$ defined by

$$u_n(x_i^n) = \overline{\rho} z \lambda_n \sum_{k=0}^i \frac{1}{\rho_k}. \qquad \square$$

## 1.8 Energies depending on second-difference quotients

We consider the case of energies

$$E_n(u) = \sum_{i=1}^{n-1} \lambda_n f\Big(\frac{u_{i+1} - 2u_i + u_{i-1}}{\lambda_n^2}\Big), \tag{70}$$

with $f$ convex and such that $c_1(|z|^p - 1) \le f(z) \le c_2(1 + |z|^p)$ $(p > 1)$.

In this case we identify the discrete function $u$ with a function in $W^{2,p}(0, L)$. Given the values $u_{i-1}, u_i, u_{i+1}$ we define the function $u$ on the interval

$$I_i^n = \Big(\frac{x_{i-1}^n + x_i^n}{2}, \frac{x_{i+1}^n + x_i^n}{2}\Big) = \Big(x_i^n - \frac{\lambda_n}{2}, x_i^n + \frac{\lambda_n}{2}\Big)$$

$(i \in \{1, \dots, n-1\})$ by

$$\begin{aligned} u(t) = & \frac{u_i + u_{i-1}}{2} + \frac{u_i - u_{i-1}}{\lambda_n}\Big(t - \frac{x_{i-1}^n + x_i^n}{2}\Big) \\ & + \frac{u_{i+1} - 2u_i + u_{i-1}}{2\lambda_n^2}\Big(t - \frac{x_{i-1}^n + x_i^n}{2}\Big)^2 \end{aligned} \tag{71}$$

Note that

$$u'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\lambda_n^2} \qquad \text{on } I_i^n,$$

and

$$u\Big(\frac{x_{i-1}^n + x_i^n}{2}\Big) = \frac{u_i + u_{i-1}}{2}, \qquad u'\Big(\frac{x_{i-1}^n + x_i^n}{2}\Big) = \frac{u_i - u_{i-1}}{\lambda_n}$$

$$u\Big(\frac{x_i^n + x_{i+1}^n}{2}\Big) = \frac{u_{i+1} + u_i}{2}, \qquad u'\Big(\frac{x_i^n + x_{i+1}^n}{2}\Big) = \frac{u_{i+1} - u_i}{\lambda_n}.$$

Finally, we set

$$u(t) = \frac{u_1 + u_0}{2} + \frac{u_1 - u_0}{\lambda_n}\Big(t - \frac{\lambda_n}{2}\Big)$$

on $(0, \lambda_n/2)$ and

$$u(t) = \frac{u_n + u_{n-1}}{2} + \frac{u_n - u_{n-1}}{\lambda_n}\Big(t - L - \frac{\lambda_n}{2}\Big)$$

on $(L - (\lambda_n/2), L)$. In this way $u \in C^1(0, L)$ and $u''$ is piecewise constant, so that $u \in W^{2,p}(0, L)$ (actually, $u \in W^{2,\infty}(0, L)$). Moreover,

$$E_n(u) = \int_0^L f(u'')\,dt. \tag{72}$$

We have the following result.

**Theorem 12.** *With the identification above, the energies $E_n$ $\Gamma$-converge as $n \to +\infty$ to the functional*

$$F(u) = \begin{cases} \displaystyle\int_{(0,l)} f(u'')\,dt & \text{if } u \in W^{2,p}(0,L) \\ +\infty & \text{otherwise} \end{cases}$$

*with respect to the convergence in $L^1(0,L)$ and weak in $W^{2,p}(0,L)$.*

*Proof.* Let $u_n \to u$ in $L^1(0,L)$ and $\sup_n E_n(u_n) < +\infty$. Then we have

$$\sup_n \Big( \int_0^L (|u_n| + |u_n''|^p)\,dt \Big) < +\infty.$$

By interpolation, we deduce that $\sup_n \|u_n\|_{W^{2,p}(0,L)} < +\infty$; hence $u_n \rightharpoonup u$ weakly in $u \in W^{2,p}(0,L)$. In particular $u_n'' \rightharpoonup u''$ in $L^p(0,L)$, so that

$$F(u) = \int_{(0,l)} f(u'')\,dt \le \liminf_n \int_{(0,l)} f(u_n'')\,dt = \liminf_n E_n(u_n).$$

If $u \in C^2([0,L])$ then, upon choosing $(u_n)_i = u(x_i^n)$ we have $u_n \to u$ and

$$E_n(u_n) = \int_{(0,l)} f(u'' + o(1))\,dt,$$

so that $\lim_n E_n(u_n) = F(u)$. For a general $u \in W^{2,p}(0,L)$ it suffices to use an approximation argument. □

## 2 Limit energies on discontinuous functions: two examples

In this chapter we begin dealing with energy density which do not satisfy a growth condition of polynomial type. We explicitly treat two model situations.

### 2.1 The Blake-Zisserman model

A finite-difference scheme proposed by Blake-Zisserman to treat signal reconstruction problems takes into account (beside other terms of 'lower order') energies defined on discrete functions of the form

$$E_n(u) = \sum_{i=1}^{n} \lambda_n \psi_n\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big), \tag{73}$$

with

$$\psi_n(z) = \min\Big\{z^2, \frac{\alpha}{\lambda_n}\Big\}, \tag{74}$$

for some $\alpha > 0$. An interesting interpretation of the energy density $\psi_n$ can be given also as relative to the energy between two neighbours in an array of material points connected by springs. In this case the springs are quadratic until a threshold, after which they bear no response to traction (broken springs).

Note that the energies above do not fit in the framework of the previous chapter, as they do not satisfy a growth condition of order $p$ from below. Note moreover that no interesting result can be obtained by taking into account the convexifications $\psi_n^{**}$ as they are trivially 0.

In this section we will treat the limit of energies modeled on $E_n$ above. We first define the proper convergence under which such energies are equi-coercive.

### Coerciveness conditions

We examine the coerciveness conditions for sequences of (piecewise-affine interpolations of) functions $(u_n)$ such that

$$\sup_n E_n(u_n) < +\infty. \tag{75}$$

For such a sequence, denote by

$$I^n = \{i \in \{1, \dots, n\} : \ |u_n(x_i^n) - u_n(x_{i-1}^n)| > \alpha\lambda_n\} \tag{76}$$

the set of indices such that

$$\psi_n(u') \neq (u')^2 \qquad \text{on } (x_{i-1}^n, x_i^n), \tag{77}$$

and by

$$S_n = \bigcup_{i \in I^n} (x_{i-1}^n, x_i^n) \tag{78}$$

the union of the corresponding intervals.

Note that we have

$$E_n(u_n) = \int_{(0,L)\setminus S_n} (u_n')^2 \, dt + \alpha \#(I^n), \tag{79}$$

so that by (75) we deduce that

$$\sup_n \#(I^n) \leq \frac{1}{\alpha} \sup_n E_n(u_n) < +\infty. \tag{80}$$

Upon extracting a subsequence, we may assume then that

$$\#(I^n) = N \qquad \text{for all } n \in \mathbb{N}, \tag{81}$$

with $N$ independent of $n$. Let $t_0^n, \dots, t_{N+1}^n$ be points in $[0, L]$ such that $t_0^n = 0$, $t_{N+1}^n = L$, $t_{i-1}^n < t_i^n$ and

$$\{t_i^n : i = 1, \dots, N+1\} = (\lambda_n I^n) \cup \{0, L\}. \tag{82}$$

Upon further extracting a subsequence we may suppose that

$$t_i^n \to t_i \in [0, L] \qquad \text{for all } i. \tag{83}$$

Denote the set of these limit points by

$$S = \{t_i : i = 0, \dots, N+1\}.$$

Let $\eta > 0$ be fixed; then for $n$ large enough we have

$$S_n = \bigcup_{i \in I^n} \Big( x_i^n - (0, \lambda_n) \Big) \subset S + (-\eta, \eta). \tag{84}$$

Hence, from (79) and (83) we deduce that

$$\begin{aligned} &\limsup_n \int_{(0,L)\setminus(S+(-\eta,\eta))} (u_n')^2\,dt \\ &\leq \sup_n \int_{(0,L)\setminus S_n} (u_n')^2\,dt \leq \sup_n E_n(u_n) < +\infty. \end{aligned} \tag{85}$$

We deduce that for every $\eta > 0$ $u_n \in W^{1,2}((0,L) \setminus (S + (-\eta, \eta)))$ and, if for every $i = 0, \dots, N$ we have

$$\liminf_n \Big( \text{ess-inf}\{|u_n(t)| : t \in (t_i^n + \eta, t_{i+1}^n - \eta)\} \Big) < +\infty, \tag{86}$$

then $(u_n)$ is weakly precompact in $W^{1,2}(t_i^n + \eta, t_{i+1}^n - \eta)$ by Poincaré's inequality. Let $u$ be its limit defined separately on each $(t_i^n + \eta, t_{i+1}^n - \eta)$. By the arbitrariness of $\eta$ we have that $u$ can be defined on $(0, L) \setminus S$, and hence a.e. on $(0, L)$. By this construction $u \in W^{1,p}_{\text{loc}}((0, L) \setminus S)$. Moreover, by (85) we deduce that for all $\eta > 0$

$$\int_{(0,L)\setminus(S+(-\eta,\eta))} (u')^2\,dt \leq \liminf_n \int_{(0,L)\setminus(S+(-\eta,\eta))} (u_n')^2\,dt \leq \sup_n E_n(u_n), \tag{87}$$

which gives a bound independent of $\eta$, so that by the arbitrariness of $\eta > 0$ we deduce that $u \in W^{1,p}((0, L) \setminus S)$.

We now introduce the following notation.

**Definition 4.** The space $P\text{-}W^{1,p}(0, L)$ of *piecewise-Sobolev functions* on $(0, L)$ is defined as the set of functions $u \in L^1(0, L)$ such that a finite set $S \subset (0, L)$ exists such that $u \in W^{1,p}((0, L) \setminus S)$. The minimal such set $S$ is called the *set of discontinuity points* or *jump set* of $u$ and denoted by $S(u)$. For such $u$ we regard the derivative $u' \in L^p(0, L)$ as defined a.e. and coinciding with its usual definition outside $S$.

We then have the following compactness result.

**Theorem 13.** *Let* $(u_n)$ *be a sequence of functions such that* $\sup_n E_n(u_n) < +\infty$ *and such that* $(u_n)$ *is bounded in measure. Then there exists a function* $u \in P\text{-}W^{1,2}(0,L)$ *such that* $u_n \to u$ *in measure. Moreover there exists a finite set* $S$ *such that* $u_n \rightharpoonup u$ *weakly in* $W^{1,p}_{\rm loc}((0,L)\setminus S)$.

*Proof.* The proof is contained in (76)–(87) above, once we remark that boundedness in measure implies (86). □

**Limit energies for nearest-neighbour interactions**

From the reasonings above we easily deduce a first convergence result.

**Theorem 14.** *Let* $E_n$ *be given by* (73)–(74)*. Then* $E_n$ *converge with respect to the convergence in measure and in* $L^1(0,L)$ *to the energy*

$$F(u) = \begin{cases} \displaystyle\int_0^L |u'|^2\,dt + \alpha\,\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{88}$$

*in* $L^1(0,L)$.

*Proof.* Let $u_n \to u$ in measure. Then by (79)–(87) it remains to show that $\#(S(u)) \le \liminf_n \#(I^n)$. This follows immediately from the facts that $S(u) \subset S$, and that, in the notation of (79)–(87), $\#(S) \le N = \lim_n \#(I^n)$.

As for the limsup inequality, it suffices to remark that if we take $u_n = u \in P\text{-}W^{1,\infty}(0,L)$ then for $n$ large $E_n(u_n) \le F(u)$. For a general $u$ we may proceed by density. □

From the lower semicontinuity properties of $\Gamma$-limits we immediately have the following corollary.

**Corollary 1.** *The functional* $F$ *in* (88) *is lower semicontinuous with respect to the convergence in measure and in* $L^1(0,L)$.

*Remark 9.* In Theorem 14 we can also consider the weak$^*$-convergence of $u_n$.

**Equivalent energies on the continuum**

The first difference that meets the eye in Theorem 14 from the theory developed for energy densities with polynomial growth is that we have two different parts of the energy densities $\psi_n$ that give rise to a bulk and a jump energy, respectively. In particular we cannot simply substitute the difference quotient by a derivative, or the function $\psi_n$ by its convexification. A counterpart of $E_n$ on the continuum is immediately obtained if we consider a different identification, other than the piecewise-affine one, for a discrete functions $u : \{x_0^n, \dots, x_n^n\} \to \mathbb{R}$: using the notation

$$I^n(u) = \{i \in \{1,\dots,n\} : \ |u(x_i^n) - u(x_{i-1}^n)| > \alpha\lambda_n\} \tag{89}$$

we may extend $u$ to the whole $(0,L)$ by setting

$$u(t) = \begin{cases} u_{i-1} + \dfrac{u_i - u_{i-1}}{\lambda_n}(t - x_i^n) & \text{if } t \in (x_{i-1}^n, x_i^n),\ i \notin I^n(u) \\ u_{i-1} & \text{if } x_{i-1}^n \le x \le x_{i-1}^n + \frac{\lambda_n}{2},\ i \in I^n(u) \\ u_i & \text{if } x_i^n - \frac{\lambda_n}{2} \le x \le x_i^n,\ i \in I^n(u). \end{cases} \tag{90}$$

Note that such extension of $u$ belongs to $P\text{-}W^{1,2}(0,L)$,

$$S(u) = \Big\{x_i^n - \frac{\lambda_n}{2} : i \in I^n(u)\Big\}, \tag{91}$$

and we have the identification

$$E_n(u) = F(u). \tag{92}$$

In this sense, $F$ is the continuous counterpart of each $E_n$.

**Limit energies for long-range interactions**

We now investigate the limit of superpositions of energies of the form (73). Let $(\rho_j)$ and $(\alpha_j)$ be given sequences of non-negative numbers. We suppose that if $\alpha_j\rho_j = 0$ then $\alpha_j = \rho_j = 0$. We define the energy densities

$$\psi_n^j(z) = \min\Big\{\rho_j z^2, \frac{\alpha_j}{\lambda_n}\Big\} \tag{93}$$

and the energies

$$E_n(u) = \sum_{j=1}^{n}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big) \tag{94}$$

**Theorem 15.** *Suppose that*

$$\rho_1 > 0, \qquad \alpha_1 > 0. \tag{95}$$

*Let $\rho, \alpha \in (0,+\infty]$ be defined by*

$$\rho = \sum_{j=1}^{\infty} \rho_j, \qquad \alpha = \sum_{j=1}^{\infty} j\alpha_j. \tag{96}$$

*Then the energies $E_n$ $\Gamma$-converge with respect to the convergence in measure and in $L^1(0,L)$ to the functional $F$ given by*

$$F(u) = \begin{cases} \rho \displaystyle\int_0^L |u'|^2\,dt + \alpha\,\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{97}$$

*in $L^1(0,L)$, where it is understood that if $\alpha = +\infty$ then $F(u) = +\infty$ if $S(u) \ne \emptyset$, and that if $\rho = +\infty$ then $F(u) = +\infty$ if $u' \ne 0$ a.e.*

*Proof.* Preliminarily note that by (95) we have that the $\Gamma$-limit (exists and) is $+\infty$ outside $P\text{-}W^{1,2}(0,L)$.

With fixed $K \in \mathbb{N}$ consider for $n \geq K$ the energies

$$E_n^K(u) = \sum_{j=1}^{K}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big), \tag{98}$$

so that

$$E_n^K(u) \leq E_n(u). \tag{99}$$

For all $j = 1,\dots,K$ and $k = 0,\dots,j-1$ let

$$E_n^{j,k}(u) = \sum_{i=0}^{[n/j]-2} \lambda_n \psi_n^j\Big(\frac{u_{k+(i+1)j}-u_{k+ij}}{j\lambda_n}\Big), \tag{100}$$

so that

$$E_n^K(u) \geq \sum_{j=1}^{K}\sum_{k=0}^{j-1} E_n^{j,k}(u). \tag{101}$$

Note that proceeding as in the proof of Theorem 14 by interpreting $E_n^{j,k}$ as an energy on the lattice $j\lambda_n\mathbb{Z} + k\lambda_n$, we easily get that $E_n^{j,k}$ $\Gamma$-converge as $n \to +\infty$ to the functional $F^j$ (independent of $k$) given by

$$F^j(u) = \begin{cases} \dfrac{\rho_j}{j}\displaystyle\int_0^L |u'|^2\,dt + \alpha_j\,\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{102}$$

We then immediately get the following liminf inequality: if $u_n \to u$ then

$$\begin{aligned} \liminf_n E_n(u_n) &\geq \liminf_n E_n^K(u_n) \\ &\geq \sum_{j=1}^{K}\sum_{k=0}^{j-1} \liminf_n E_n^{j,k}(u_n) \\ &\geq \sum_{j=1}^{K} jF^j(u) = \sum_{j=1}^{K}\int_0^L |u'|^2\,dt + \sum_{j=1}^{K} j\alpha_j\#(S(u)). \end{aligned}$$

The desired inequality is obtained by letting $K \to +\infty$, and using the Monotone Convergence Theorem.

Let now $u \in P\text{-}W^{1,2}(0,L)$ be such that $F(u) < +\infty$. Consider first the case $\alpha < +\infty$, $\rho < +\infty$. By a density argument it suffices to consider the case $u \in P\text{-}W^{1,\infty}(0,L)$. In this case we can choose $u_n = u$, and note that

$$\limsup_n E_n(u_n) \leq \sum_{j=1}^{K} j \lim_n F^j(u_n) + c \sum_{j=K+1}^{\infty}\Big(\rho_j\|u'\|_\infty^2 + j\alpha_j\#(S(u))\Big).$$

In the case when $\alpha = +\infty$ it suffices to compare with the convex case as $\psi_n^j(z) \leq \rho_j z^2$. When $\rho = +\infty$ $F$ is finite only on piecewise-constant $u$, for which we take $u_n = u$ and the computation is straightforward. □

**Boundary-value problems**

In contrast to what happened to functionals with limits defined on Sobolev spaces, the coerciveness conditions at our disposal do not guarantee that minimizers satisfying some boundary conditions converge to a minimizer satisfying the same boundary condition. We have thus to relax the notion of boundary values.

We consider boundary-value problems given in two ways.

(I) **Interaction at the boundary**: we fix two values $U_0$ and $U_L$ and add to the energy $E_n$ the constraint $u(0) = U_0$, $u(L) = U_L$;

(II) **Interaction through the boundary**: we fix $\phi : \mathbb{R} \to \mathbb{R}$ and add to $E_n$ the 'boundary-value term'

$$B_n(u) = \sum_{j=1}^{n+K_n} \sum_{i=\max\{-j,-K_n\}}^{-1} \lambda_n \psi_n^j\Big(\frac{u(x_{i+j}^n) - \phi(x_i^n)}{j\lambda_n}\Big) + \sum_{j=1}^{n+K_n} \sum_{i=n+1}^{\min\{n+j,n+K_n\}} \lambda_n \psi_n^j\Big(\frac{\phi(x_i^n) - u(x_{i-j}^n)}{j\lambda_n}\Big),$$

which corresponds to setting $u = \phi$ outside $[0, L]$ and to considering the energy of this extension on an enlarged interval with an addition of a 'layer' of size $K_n\lambda_n$ on both sides of the interval.

We first treat the case (II). For the sake of simplicity we consider the case when $\rho_j = 0$ for $j > K$, and we choose $K_n = K$. In this case our energy $E_n + B_n$ can be written as

$$\tilde{E}_n(u) = \sum_{j=1}^{K} \sum_{i=-j}^{n} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{103}$$

with the constraint

$$u_i = \phi(x_n^i) \qquad \text{for } i \in \{-K, \dots, -1\} \cup \{n+1, \dots, n+K\} \tag{104}$$

**Theorem 16.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $\tilde{E}_n$ be given by (103)–(104). Then $\tilde{E}_n$ $\Gamma$- converges to the functional $\tilde{F}$ given by*

$$\tilde{F}(u) = \begin{cases} \rho \displaystyle\int_0^L |u'|^2\,dt + \alpha\#(\{x \in [0,L] : u(x+) \neq u(x-)\}) & \\ & \text{if } u \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise,} \end{cases} \tag{105}$$

*where*

$$\rho = \sum_{j=1}^{K} \rho_j, \qquad \alpha = \sum_{j=1}^{K} \alpha_j,$$

*and we have set*

$$u(0-) = \phi(0), \qquad u(L+) = \phi(L). \tag{106}$$

*Proof.* Let $E_n(v,(-L,2L))$ be defined by

$$E_n(v,(-L,2L)) = \sum_{j=1}^{K}\sum_{i=-n}^{2n-j} \lambda_n \psi_n^j\Big(\frac{v_{i+j}-v_i}{j\lambda_n}\Big).$$

the choice of the interval $(-L,2L)$ has been done only for convenience of notation; indeed any open interval containing $[0,L]$ would do. By the previous results $E_n(\cdot,(-L,2L))$ $\Gamma$-converges to the functional $F(\cdot,(-L,2L))$ with domain $P\text{-}W^{1,2}(-L,2L)$ and defined there by

$$F(v,(-L,2L)) = \rho\int_{-L}^{2L} |v'|^2\,dt + \alpha\#(S(v)).$$

Let $u_n \to u$ in measure on $(0,L)$. Let $v_n$ be defined by

$$v_n(x_i^n) = \begin{cases} \phi(0) & \text{if } i<0\\ u_n(x_i^n) & \text{if } 0\le i\le n\\ \phi(L) & \text{if } i>n,\end{cases}$$

and similarly define also $v$. Note that $v_n \to v$ in measure on $(-L,2L)$. We then have

$$\begin{aligned}\liminf_n \tilde{E}_n(u_n) &= \liminf_n E_n(v_n,(-L,2L))\\ &\ge F(v,(-L,2L)) = F(u).\end{aligned}$$

To obtain the limsup inequality it suffices to take $u_n = u$. □

In the case (I) we treat arbitrarily long-range interactions.

**Theorem 17.** *Let $E_n$ be given by* (94) *and let $\tilde{E}_n$ be given by*

$$\tilde{E}_n(u) = \begin{cases} E_n(u) & \text{if } u(0)=U_0 \text{ and } u(L)=U_L\\ +\infty & \text{otherwise.}\end{cases} \tag{107}$$

*Then $\tilde{E}_n$ $\Gamma$- converges to the functional $\tilde{F}$ given by*

$$\tilde{F}(u) = \begin{cases} \rho\displaystyle\int_0^L |u'|^2\,dt + \alpha\#(S(u)) + \alpha_0\#(\{x\in\{0,L\} : u(x+)\ne u(x-)\}) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u\in P\text{-}W^{1,2}(0,L) & \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise,} & \end{cases} \tag{108}$$

*where*

$$\rho = \sum_{j=1}^{\infty}\rho_j, \qquad \alpha = \sum_{j=1}^{\infty} j\alpha_j, \qquad \alpha_0 = \sum_{j=1}^{\infty}\alpha_j, \tag{109}$$

*and we have set*

$$u(0-) = U_0, \qquad u(L+) = U_L. \tag{110}$$

*Proof.* We begin with the case $j = 1$ and with a boundary condition on only one side (*e.g.* at 0). Consider the functional

$$E_n^0(u) = \begin{cases} E_n(u) & \text{if } u(0) = U_0 \\ +\infty & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 16 we can write

$$E_n^0(u) = E_n(v, (-L, L)),$$

where

$$v(x_i^n) = \begin{cases} U_0 & \text{if } i \leq 0 \\ u(x_i^n) & \text{if } 0 < i \leq n \end{cases}$$

and

$$E_n(v, (-L, L)) = \sum_{i=-n}^{n-1} \lambda_n \psi_n^1 \Big( \frac{u_{i+1} - u_i}{\lambda_n} \Big).$$

If $u_n \to u$ we then obtain

$$\liminf_n E_n(u_n) \geq \rho_1 \int_0^L |u'|^2\, dt + \alpha_1 \#(S(u)) + \alpha_1(1 - \chi_0(u(0+) - U_0)).$$

The limsup inequality is immediately obtained by taking $u_n = u$.

In the same way we treat the boundary condition at $L$ and the boundary conditions at both sides.

With fixed $K$ can repeat the same reasoning as above for all $j \in \{1, \dots, K\}$ such that $\alpha_j \rho_j \neq 0$ (otherwise the limit is trivial) and obtain that the $\Gamma$-limit of

$$E_n^{0,j}(u) = \begin{cases} \displaystyle\sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big( \frac{u_{i+j} - u_i}{\lambda_n} \Big) & \text{if } u_0 = U_0 \\ +\infty & \text{otherwise} \end{cases}$$

as $n \to +\infty$ is given by

$$\rho_j \int_0^L |u'|^2\, dt + j\alpha_j \#(S(u)) + \alpha_j(1 - \chi_0(u(0+) - U_0)).$$

Symmetrically we can treat the case $u_n = U_L$. The case of boundary condition on both sides gives that the $\Gamma$-limit of

$$E_n^{0,L,j}(u) = \begin{cases} \displaystyle\sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big( \frac{u_{i+j} - u_i}{\lambda_n} \Big) & \text{if } u(0) = U_0 \text{ and } u_n = U_L \\ +\infty & \text{otherwise} \end{cases}$$

is

$$\rho_j \int_0^L |u'|^2\,dt + j\alpha_j \#(S(u)) \\ +\alpha_j(1-\chi_0(u(0+)-U_0)) + \alpha_j(1-\chi_0(u(L-)-U_L)).$$

Summing up these considerations we obtain that for all $K$

$$\Gamma\text{-}\liminf_n \tilde{E}_n \geq \rho^K \int_0^L |u'|^2\,dt \\ +\alpha^K \#(S(u)) + \alpha_0^K \#(\{x \in \{0,L\} : u(x+) \neq u(x-)\}),$$

where $\rho^K = \sum_{j=1}^K \rho_j$, $\alpha^K = \sum_{j=1}^K j\alpha_j$ and $\alpha_0^K = \sum_{j=1}^K \alpha_j$. The liminf inequality is obtained by taking the supremum in $K$.

The upper inequality is obtained by taking $u_n = u$. □

*Remark 10.* Note that we may have $\alpha = +\infty$ but $\alpha_0 < +\infty$, in which case $\tilde{F}$ is finite only on $W^{1,2}(0,L)$ but may be finite also on functions not matching the boundary conditions.

### Homogenization

We only treat the case of nearest-neighbour interactions. Let $i \mapsto \rho_i$ and $i \mapsto \alpha_i$ define $M$-periodic functions $\mathbb{Z} \to \mathbb{R}$:

$$\rho_{i+M} = \rho_i, \qquad \alpha_{i+M} = \alpha_i \qquad \text{for all } i.$$

The energies we consider take the form

$$E_n(u) = \sum_{i=0}^{n-1} \min\Big\{\lambda_n \rho_i \Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)^2, \alpha_i\Big\}. \tag{111}$$

**Theorem 18.** *The energies $E_n$ $\Gamma$-converge to the energy defined by*

$$F(u) = \begin{cases} \overline{\rho} \displaystyle\int_0^L |u'|^2\,dt + \overline{\alpha}\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise,} \end{cases}$$

*where*

$$\overline{\rho} = M\Big(\sum_{i=1}^M \frac{1}{\rho_i}\Big)^{-1}, \qquad \overline{\alpha} = \min_i \alpha_i.$$

*Proof.* By following the proof of Theorem 11 we immediately obtain the liminf inequality.

In order to construct a recovery sequence for the $\Gamma$-limsup, let $u \in P\text{-}W^{1,2}(0,L)$ and define $v(t) = u(0+) + \int_0^t u'(s)\,ds$. Let $v_n$ be a recovery sequence for the $\Gamma$-limit in Theorem 11 computed at $v$. Let $k \in \{0,\dots,M-1\}$

be such that $\overline{\alpha} = \alpha_k$. Then for all $t \in S(u)$ let $j_n(t) \equiv k \bmod M$ be such that $|x^n_{j_n(t)} - t| \le M\lambda_n$. Then the functions

$$u_n(x^n_i) = v_n(x^n_i) + \sum_{t\in S(u):\ j_n(t)\le i} (u(t+) - u(t-))$$

define a recovery sequence for $u$. □

**Non-local limits**

For all $n \in \mathbb{N}$ let $\rho_n : j\mathbb{Z} \to [0, +\infty)$. We consider the long-range discrete energies of Blake-Zisserman type

$$E_n(u) = \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L]\\ x\neq y}} \rho_n(x-y)\, \Psi_n\Big(\frac{u(x)-u(y)}{x-y}\Big) \tag{112}$$

defined for $u : \lambda_n\mathbb{Z} \to \mathbb{R}$, where

$$\Psi_n(z) = \min\{\lambda_n z^2, 1\}.$$

We make the same assumptions on $(\rho_n)$ as in Section 1.7.

**Theorem 19.** *If conditions* (H1) *and* (H2) *in Section* 1.7 *hold, then there exist a subsequence (not relabelled), a Radon measure $\mu$ on $\mathbb{R}^2$, a constant $c_1 > 0$ and an even subadditive and lower semicontinuous function $\varphi : \mathbb{R} \to [0, +\infty]$ such that the energies $E_n$ $\Gamma$-converge to the energy $F$ defined on* $\mathrm{L}^1(0, L)$ *by*

$$F(u) = \begin{cases} c_1 \displaystyle\int_{(0,L)} |u'|^2\, dt + \sum_{S(u)} \varphi([u]) + \int_{(0,L)^2} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y) \\ \qquad\qquad\qquad\qquad\qquad \text{if } u \text{ is piecewise } W^{1,2} \text{ on } [0,L] \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{113}$$

*where $S(u)$ denotes the set of discontinuity points for $u$ and $[u](t) = u(t+) - u(t-)$ is the jump of $u$ at $t$. The measure $\mu$ and $c_1$ are given by* (66) *and* (65)*, respectively, and the function $\varphi$ is given by the* discrete phase-transition energy density formula

$$\varphi(z) = \liminf_{m\to+\infty} \inf_{|w|<|z|} \lim_n \min\Big\{ \sum_{\substack{j,k\in\mathbb{Z},\ j\neq k\\ -2/m\lambda_n\le j,k\le 2/m\lambda_n}} \rho_n(\lambda_n(j-k))\Psi_n\Big(\frac{u(j)-u(k)}{\lambda_n(j-k)}\Big) : \\ u:\mathbb{Z}\to\mathbb{R},\ u(j)=0 \text{ if } j < -\frac{1}{m\lambda_n},\ u(j)=w \text{ if } j > \frac{1}{m\lambda_n}\Big\} \tag{114}$$

*for $z \in \mathbb{R}$.*

*Remark 11.* (i) Since $\varphi$ is subadditive, and it is also non decreasing on $[0,+\infty)$ and even, we have that either it is finite everywhere or $\varphi(z)=+\infty$ for all $z\neq 0$. In the latter case jumps are prohibited and the domain of $F$ is indeed $W^{1,2}(0,L)$.

(ii) We will show below that the function $\varphi$ may be not constant, in contrast with the case when $\rho_n(z)=\rho(z/\lambda_n)$ for a fixed $\rho$.

*Proof.* With fixed $m,n\in\mathbb{N}$ the minimum value in (114) defines an even function of $w$ which is non-decreasing on $[0,+\infty)$; hence, by Helly's Theorem there exists a sequence (not relabeled) $\{\lambda_n\}$ such that these minimum values converge for all $w$ and for all $m$. Hence, we can assume, upon passing to this subsequence $\{\lambda_n\}$, that the function $\varphi$ is well defined. Upon passing to a further subsequence we may also assume that the measures $\mu_n$ in Remark 8 converge to $\mu_0$. Then, $\mu$ and $c_1$ given by (66) and (65) are well defined as well. Hence, it suffices to prove the representation for the $\Gamma$-limit along this sequence, since the subadditivity and lower semicontinuity of $\varphi$ are necessary conditions for the lower semicontinuity of $F$.

We begin by proving the liminf inequality. Let $u_n\to u$ in $\mathrm{L}^1(0,L)$ be such that $\sup_n E_n(u_n)<+\infty$. By hypothesis (H1), if we set

$$S^n=\{x\in\lambda_n\mathbb{Z}:\ |u(x+\lambda_n)-u(x)|^2>1/\lambda_n\},$$

then $\#S^n$ is equibounded, and, upon extracting a subsequence, we can suppose that $S^n=\{x_j^n:\ j=1,\dots,N\}$ with $N$ independent of $n$ $x_1^n<x_2^n<\dots<x_N^n$ and $x_j^n\to t_j$ for all $j$. Set $S=\{t_j\}\subset[a,b]$. If $\{x^n_{M_1}\},\dots,\{x^n_{M_2}\}$ are the sequences converging to $t\in S$ then $u_n(x^n_{M_1})\to u(t-)$ and $u_n(x^n_{M_2}+\lambda_n)\to u(t+)$. Furthermore, the sequence $u_n$ converges locally weakly in $W^{1,2}((0,L)\setminus S)$.

For all $\eta>0$ let $S_\eta=\{t\in\mathbb{R}:\ \mathrm{dist}\,(t,S)<\eta\}$; set also $\Delta_\eta=\{(x,y)\in\mathbb{R}^2:\ |x-y|>\eta\}$. Note that the convergence

$$\frac{u_n(x)-u_n(y)}{x-y}\ \longrightarrow\ \frac{u(x)-u(y)}{x-y}$$

as $n\to\infty$ is uniform on $(0,L)^2\setminus(S^2_\eta\cup\Delta_\eta)$.

With fixed $m\in\mathbb{N}$, we have the inequality

$$E_n(u_n)\geq\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L]\cap S_{4/m}\\ |x-y|\leq 4/m,\ x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)$$
$$+\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L]\setminus S_{4/m}\\ |x-y|\leq 4/m,\, x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)$$

$$+\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[0,L]\\|x-y|>4/m}}\rho_n(x-y)\Psi_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)$$
$$=: I_n^1(u_n)+I_n^2(u_n)+I_n^3(u_n). \tag{115}$$

The terms $I_n^2(u_n)$ and $I_n^3(u_n)$ can be dealt with as in Section 1.7. We now deal with $I_n^1(u_n)$. We first note that

$$I_n^1(u_n)\geq\sum_{t\in S(u)}\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[t-(2/m),t+(2/m)]\\x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big). \tag{116}$$

We use the notation introduced above for the sets $S^n$ and $S$: let $t_j\in S(u)$ with corresponding sequences $\{x^n_{M_1}\},\dots,\{x^n_{M_2}\}$ converging to $t_j$. We can suppose, up to a translation and reflection argument, that $[u](t_j)>0$, that

$$\max\{u_n(x):\ x\in\lambda_n\mathbb{Z},\ t_j-(2/m)\leq x\leq x^n_{M_1}\}=0$$

and that

$$\min\{u_n(x):\ x\in\lambda_n\mathbb{Z},\ x^n_{M_2}+\lambda_n\leq x\leq t_j+(2/m)\}=z_n,$$

with $z_n\to[u](t_j)$. We then have

$$\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)$$
$$\geq\min\Big\{\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{v(x)-v(y)}{x-y}\Big):$$
$$v(x^n_{M_1})=u_n(x^n_{M_1}),\ v(x^n_{M_2}+\lambda_n)=u_n(x^n_{M_2}+\lambda_n)\Big\}$$
$$\geq\min\Big\{\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{v(x)-v(y)}{x-y}\Big):$$
$$v(x)=0\text{ if }x\leq x^n_{M_1},\ v(x)=z_n\text{ if }x\geq x^n_{M_2}+\lambda_n\Big\}$$
$$\geq\min\Big\{\sum_{\substack{x,y\in\lambda_n\mathbb{Z}\cap[t_j-(2/m),t_j+(2/m)]\\x\neq y}}\rho_n(x-y)\Psi_n\Big(\frac{v(x)-v(y)}{x-y}\Big):$$
$$v(x)=0\text{ if }t_j-\frac{2}{m}\leq x\leq t_j-\frac{1}{m},\ v(x)=z_n\text{ if }t_j+\frac{1}{m}\leq x\leq t_j+\frac{2}{m}\Big\}$$
$$=\min\Big\{\sum_{\substack{j,k\in\mathbb{Z}\cap[-2/(m\lambda_n),2/(m\lambda_n)]\\j\neq k}}\rho_n(\lambda_n(j-k))\Psi_n\Big(\frac{v(j)-v(k)}{\lambda_n(j-k)}\Big):$$
$$v(j)=0\text{ if }-\frac{2}{m\lambda_n}\leq j\leq-\frac{1}{m\lambda_n},\ v(j)=z_n\text{ if }\frac{1}{m\lambda_n}\leq j\leq\frac{2}{m\lambda_n}\Big\}. \tag{117}$$

Note that we have used the fact that $\Psi_n$ in non decreasing on $(0,+\infty)$ so that our functionals decrease by truncation (namely, when we substitute $v$ by $(v\vee 0)\wedge z_n$). By taking (114) into account and summing up for $t_j\in S(u)$, we obtain

$$\liminf_n I^1_n(u_n)\geq \sum_{t\in S(u)}\varphi([u](t))+o(1) \tag{118}$$

as $m\to+\infty$.

By summing up this inequality to those obtained in Section 1.7 and letting $m\to+\infty$ we eventually get

$$\liminf_n E_n(u_n)\geq c_1\int_0^L |u'|^2\,dt+\sum_{S(u)}\varphi([u])+\int_{(0,L)^2}\Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

We now prove the limsup inequality. It suffices to show it for piecewise-affine functions, since this set is strongly dense in the space of piecewise $W^{1,2}$ functions. We explicitly treat the case when $(0,L)$ is replaced by $(-1,1)$ and

$$u(t)=\begin{cases}\alpha t & \text{if } t<0\\ \beta t+z & \text{if } t>0\end{cases}$$

only, as the general case easily follows by repeating the construction we propose locally in the neighbourhood of each point in $S(u)$. It is not restrictive to suppose that $z>0$, by a reflection argument, and that $\varphi(z)<+\infty$, otherwise there is nothing to prove.

Let $\eta>0$, let $m\in\mathbb{N}$ with $0<1/m<\eta$ and let $z-(1/m)<z_m<z$ be such that

$$\varphi(z)\geq \lim_n \min\Big\{\sum_{x,y\in\mathbb{Z},-2/(m\lambda_n)\leq j,k\leq 2/(m\lambda_n)}\rho_n(\lambda_n(j-k))\Psi_n\Big(\frac{u(j)-u(k)}{\lambda_n(j-k)}\Big)\ :$$
$$u:\mathbb{Z}\to\mathbb{R},\ u(j)=0 \text{ if } j<-\frac{1}{m\lambda_n},\ u(j)=z_m \text{ if } j>\frac{1}{m\lambda_n}\Big\}-\eta. \tag{119}$$

Then there exist functions $v^m_n:\lambda_n\mathbb{Z}\to\mathbb{R}$ such that $v^m_n(x)=0$ for $x<-1/m$, $v^m_n(x)=z_m$ for $x>T$, $0\leq v^m_n\leq z_m$ and

$$\lim_n \sum_{\substack{x,y\in\lambda_n\mathbb{Z}\\ -(2/m)\leq x,y\leq(2/m)}}\rho_n(x-y)\,\Psi_n\Big(\frac{v^m_n(x)-v^m_n(y)}{x-y}\Big)\leq\varphi(z)+\eta.$$

We set

$$u^m_n(t)=\begin{cases}u(t+(2/m)) & \text{if } t<-2/m\\ v^m_n(t) & \text{if } -2/m\leq t\leq 2/m\\ u(t-(2/m)) & \text{if } t>2/m.\end{cases}$$

Note that $u_n^m \to u^m$ in $\mathrm{L}^1((-1,1)\setminus[-1/m,1/m])$ as $n\to\infty$, where

$$u^m(t) = \begin{cases} u(t+(2/m)) & \text{if } t< -2/m \\ 0 & \text{if } -2/m \le -1/m \\ z & \text{if } 1/m < t \le 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

We can then easily estimate

$$\begin{aligned} &\limsup_n E_n(u_n^m) \\ &\le \limsup_n \sum_{\substack{x,y\in\lambda_n\mathbb{Z}, x\ne y\\ -2/m\le x,y\le 2/m}} \rho_n(x-y)\,\Psi_n\Big(\frac{v_n^m(x)-v_n^m(y)}{x-y}\Big) \\ &\quad+\limsup_n \int_{(0,L)^2\setminus\Delta_{2/m}} \rho_n(x-y)\,\frac{1}{\lambda_n}\Psi_n\Big(\frac{u_n^m(x)-u_n^m(y)}{x-y}\Big)d\mu_n \\ &\quad+\limsup_n \sum_{x,y\in\lambda_n\mathbb{Z}\cap[0,L],\ x,y<-1/m,\ |x-y|\le 2/m} \rho_n(x-y)\,\Psi_n\Big(\frac{u_n^m(x)-u_n^m(y)}{x-y}\Big) \\ &\quad+\limsup_n \sum_{x,y\in\lambda_n\mathbb{Z}\cap[0,L],\ x,y>1/m,\ |x-y|\le 2/m} \rho_n(x-y)\,\Psi_n\Big(\frac{u_n^m(x)-u_n^m(y)}{x-y}\Big) \\ &\le \varphi(z)+\eta+\int_{(0,L)^2}\Big(\frac{u^m(x)-u^m(y)}{x-y}\Big)^2 d\mu + c_1\int_{(0,L)}|u'|^2\,dt + o(1) \end{aligned}$$

as $m\to+\infty$. Note that we have used the fact that by (64) the limit measure $\mu$ does not charge $\partial(0,L)^2$. By choosing $m=m(\lambda_n)$ with $m(\lambda_n)\to+\infty$ as $n\to\infty$, and setting $u_n = u_n^{m(\lambda_n)}$ we obtain the desired inequality. □

In the following examples for simplicity we drop the hypothesis that $\rho_n$ is even.

*Example 1.* The function $\varphi$ is not always constant. As an example, take

$$\rho_n(z) = \begin{cases} 1 & \text{if } z=\lambda_n \\ \sqrt{\lambda_n} & \text{if } z = \lambda_n[1/\sqrt{\lambda_n}] \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily seen that the minimum for the problem defining $\varphi$ is achieved on the function $v = z\chi_{(0,+\infty)}$, which gives

$$\varphi(z) = \min\{1+z^2, 2\}.$$

Note that in this case the $\Gamma$-limit is

$$\int_{(0,L)} |u'|^2\,dt + \sum_{S(u)} \varphi([u]),$$

which is local, but not with $\varphi$ constant.

*Example 2.* If we take

$$\rho_n(z) = \begin{cases} 1 & \text{if } z = \lambda_n \\ 4\sqrt{\lambda_n} & \text{if } z = \lambda_n[1/\sqrt{\lambda_n}] \\ 0 & \text{otherwise} \end{cases}$$

then by using the (discretization of) $v = z\chi_{(0,+\infty)}$ as a test function we deduce the estimate

$$\varphi(z) \leq \min\{1 + 4z^2, 5\}.$$

Since the right hand side is not subadditive, which is a necessary condition for lower semicontinuity, we deduce that the minimum in the definition of $\varphi$ is obtained by using more than one 'discontinuity'.

*Remark 12.* By the density of the sums of Dirac deltas in the space of Radon measures on the real line, in the limit functional we may obtain any measure $\mu$ satisfying the invariance property

$$\mu(A) = \mu(A + t(e_1 + e_2))$$

for all Borel set $A$ and $t \in \mathbb{R}$.

*Remark 13.* In the formula defining $\varphi$ we cannot substitute the limit of minimum problems on $[-2/(m\lambda_n), 2/(m\lambda_n)]$ by a transition problem on the whole discrete line. In fact, if we take

$$\rho_n(x) = \begin{cases} 1 & \text{if } x = \lambda_n \text{ or } x = \lambda_n[1/\lambda_n] \\ 0 & \text{otherwise,} \end{cases}$$

then the two results are different.

*Example 3.* By again taking $\rho_n$ as in the previous remark, we check that in this case $\mu = (1/\sqrt{2})\mathcal{H}^1 \llcorner (r_1 \cup r_{-1})$, where $r_i = \{x - y = i\}$.

### 2.2 Second-difference interactions

We may consider the same energies as in (73) but depending on second-difference quotients; *i.e.*, of the form

$$E_n(u) = \sum_{i=1}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1} - 2u_i + u_{i-1}}{\lambda_n^2}\Big), \tag{120}$$

with $\psi_n$ as in (74).

To understand the effect of the non-convexity threshold in $\psi_n$, first consider the case of a *jump*: $u_i = 0$ for $i \leq i_0$ and $u_i = z \neq 0$ for $i > i_0$. Then we have (for $n$ large)

$$\lambda_n\psi_n\Big(\frac{u_{i+1}-2u_i+u_{i-1}}{\lambda_n^2}\Big)=\begin{cases}\alpha & \text{if } i=i_0 \text{ or } i=i_0+1\\ 0 & \text{otherwise,}\end{cases}$$

so that $E_n(u)=2\alpha$. If instead we have a *crease*; *e.g.*, $u_i=0$ for $i\leq i_0$ and $u_i=c\lambda_n(i-i_0)$ $c\neq 0$ for $i>i_0$, then we have

$$\lambda_n\psi_n\Big(\frac{u_{i+1}-2u_i+u_{i-1}}{\lambda_n^2}\Big)=\begin{cases}\alpha & \text{if } i=i_0\\ 0 & \text{otherwise,}\end{cases}$$

so that $E_n(u)=2\alpha$. Mixing the arguments of Theorems 12 and 14 we obtain the following result (see [15]).

**Theorem 20 (jumps and crease limit energies).** *Let $E_n$ be defined above. Then the domain of the $\Gamma$-limit $F$ is the space of (possibly discontinuous) piecewise-$W^{2,2}$ functions, on which it takes the form*

$$F(u)=\int_0^L |u''|^2\,dt+2\alpha\#(S(u))+\alpha\#(S(u')\setminus S(u)).$$

Note that the factor 2 in front of the ‘jump energy’ may be justified by relaxation arguments, as a jump point can be approximated by two ‘crease points’ (see [12] for the conditions for the lower-semicontinuity of this type of energies).

### 2.3 Lennard-Jones potentials

We now consider a function $J:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ modeling inter-atomic interactions, with the properties

(i) $J(z)=+\infty$ if $z\leq 0$;
(ii) $J$ is smooth on $(0,+\infty)$;
(iii) $\lim_{z\to 0}J(z)=+\infty$.
(iv) $J$ is strictly convex on $(0,T)$;
(v) $J$ is strictly concave on $(T,+\infty)$;
(vi) $\lim_{z\to+\infty}J(z)=0$.

Our assumptions are modeled on

$$J(z)=\frac{k_1}{z^{12}}-\frac{k_2}{z^6} \tag{121}$$

for $z>0$ All these conditions can be relaxed, and we refer to the general treatment in the next chapter for weaker assumptions.

Note that hypotheses (ii)–(vi) imply that there exists a unique minimum point, which we denote by $M\in(0,T)$, and that $\min J<0$.

The energy we will consider are, with fixed $K\geq 1$,

$$E_n(u)=\sum_{j=1}^{K}\sum_{i=0}^{n-j}\lambda_n J\Big(\frac{u_{i+j}-u_i}{\lambda_n}\Big). \tag{122}$$

Note the scaling in the argument of $J$; in terms of the general form considered in the previous chapter, we have $\psi_n^j(z)=J(jz)$.

### Coerciveness conditions

Note that $E_n(u)$ is finite only if $u$ is strictly increasing; hence, we can use the strong compactness properties of increasing functions. In particular, if $(u_n)$ is a sequence of functions locally equi-bounded on $(0, L)$ then there exists a subsequence converging in $L^1_{\rm loc}(0, L)$, and if all functions are equi-bounded (*e.g.*, if they satisfy some fixed boundary conditions) then there exists a subsequence converging in $L^1(0, L)$ (actually, in $L^p(0, L)$ for all $p < \infty$). Note moreover that, by Helly's Theorem, upon passing to a further subsequence we can obtain convergence *everywhere* on $(0, L)$.

### Nearest-neighbour interactions

We begin by treating the case $K = 1$; *i.e.*,

$$E_n(u) = \sum_{i=1}^{n} \lambda_n J\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big). \tag{123}$$

It is easily seen that the $\Gamma$-limit is finite on *all* increasing functions. However, deferring the general treatment to the next chapter, we characterize the limit only on $P\text{-}W^{1,1}(0, L)$.

**Theorem 21.** *The energies $E_n$ $\Gamma$-converge on $P\text{-}W^{1,1}(0, L)$ with respect to the $L^1(0, L)$ convergence, to the functional $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dt & \text{if } u(t+) > u(t-) \text{ on } S(u) \\ +\infty & \text{otherwise} \end{cases} \tag{124}$$

*on $P\text{-}W^{1,1}(0, L)$, where*

$$\psi(z) = J^{**}(z) = \begin{cases} J(z) & \text{if } z \le M \\ \min J & \text{if } z > M. \end{cases}$$

*is the convex envelope of $J$.*

Note that the condition $u(t+) > u(t-)$ on $S(u)$ translates the fact that $F$ must be finite only on increasing functions.

*Proof.* Note preliminarily that $F$ will be finite only on increasing functions so that we need to identify it only on functions $u$ satisfying $u(t+) > u(t-)$ on $S(u)$.

Recall that the functional

$$u \mapsto \int_{(a,b)} \psi(u')\,dt$$

is lower semicontinuous on $W^{1,1}(a,b)$ with respect to the $L^1(a,b)$ convergence. Let $u \in P\text{-}W^{1,1}(0,L)$ and write

$$(0,L) \setminus S(u) = \bigcup_{k=1}^{N} (y_{k-1}, y_k), \tag{125}$$

where $0 = y_0 < \cdots < y_N = L$. Let $u_n \to u$ in $L^1(0,L)$ and $E_n(u_n) < +\infty$ for all $n$. Then we have

$$F(u) = \sum_{k=1}^{N} \int_{(y_{k-1},y_k)} \psi(u')\,dt \le \sum_{k=1}^{N} \liminf_n \int_{(y_{k-1},y_k)} \psi(u_n')\,dt$$
$$\le \liminf_n \int_{(0,L)} \psi(u_n')\,dt \le \liminf_n \int_{(0,L)} J(u_n')\,dt = \liminf_n E_n(u_n).$$

Conversely, let $u \in P\text{-}W^{1,1}(0,L)$ with $u(t+) > u(t-)$ on $S(u)$, and let $u_n = u$. Then it is easily seen that

$$\lim_n E_n(u_n) = \int_{(0,L)} J(u')\,dt,$$

so that

$$\Gamma\text{-}\limsup_n E_n(u) \le \int_{(0,L)} J(u')\,dt.$$

Now, using the notation (125), let $(u_j^k)_j$ converge to $u$ weakly in $W^{1,1}(y_{k-1}, y_k)$ (and hence also uniformly) and satisfy

$$\lim_j \int_{(y_{k-1},y_k)} J((u_j^k)')\,dt = \int_{(y_{k-1},y_k)} \psi(u')\,dt.$$

Note that for $j$ sufficiently large the function $u_j$ defined by

$$u_j = u_j^k \text{ on } (y_{k-1}, y_k)$$

satisfies $u_j(t+) > u_j(t-)$ on $S(u_j) = S(u)$ and $u_j \to u$ in $L^1(0,L)$, so that, by the lower semicontinuity of the $\Gamma$-limsup we have

$$\Gamma\text{-}\limsup_n E_n(u) \le \liminf_j \Gamma\text{-}\limsup_n E_n(u_j) \le \liminf_j \int_{(0,L)} J(u_j')\,dt$$
$$= \sum_{k=1}^{N} \lim_j \int_{(y_{k-1},y_k)} J((u_j^k)')\,dt$$
$$= \sum_{k=1}^{N} \int_{(y_{k-1},y_k)} \psi(u')\,dt = F(u),$$

and the proof is concluded. $\square$

### Higher-order behaviour of nearest-neighbour interactions

Note that minimum problems involving the limit functional $F$ present a completely different behaviour depending on whether the (trivial) convexification of $J$ is taken into account or not. Consider for example the simple minimum problem

$$m = \min\Big\{F(u) : u(0) = 0,\ u(L) = h\Big\}, \tag{126}$$

with $h > 0$. Then we have:

(*compression*) if $h \leq ML$ then the minimum $m = L\,J(h/L)$ is achieved only by the linear function $u(x) = hx/L$. Note that the minimizer has no jump;

(*tension*) if $h > ML$ then the minimum $m = L \min J$ is achieved by all functions $u \in P\text{-}W^{1,1}(0,L)$ such that $u' \geq M$ a.e. Note in particular that we can exhibit minimizers with an arbitrary number of jumps.

In this second case, hence, very little information on the behaviour of the minimizers of

$$m_n = \min\Big\{E_n(u) : u(0) = 0,\ u(L) = h\Big\}, \tag{127}$$

can be drawn from the study of the corresponding problem (126) for the $\Gamma$-limit.

To improve this description, we note now that minimizers of $m_n$ also minimize

$$m_n^{(1)} = \min\Big\{\frac{E_n(u) - L\,\min J}{\lambda_n} : u(0) = 0,\ u(L) = h\Big\}. \tag{128}$$

The choice of the scaling $\lambda_n$ is suggested by the fact that, choosing $\overline{u}_n = \overline{u}$, where $\overline{u} \in P\text{-}W^{1,1}(0,L)$ is any function with $u' = M$ a.e. we have $E_n(\overline{u}_n) \leq L\,\min J + c\lambda_n$. We are then lead to studying the $\Gamma$-limit of the scaled functions

$$E_n^{(1)}(u) = \frac{E_n(u) - L\,\min J}{\lambda_n}. \tag{129}$$

**Theorem 22.** *The functionals $E_n^{(1)}$ $\Gamma$-converge with respect to the $L^1(0,L)$ convergence to the functional $F^{(1)}$ given by*

$$F^{(1)}(u) = \begin{cases} -\min J\,\#(S(u)) & \text{if } u \in P\text{-}W^{1,\infty}(0,L),\ u(t+) > u(t-) \text{ on } S(u) \\ & \text{and } u' = M \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases} \tag{130}$$

*on $L^1(0,L)$ .*

*Proof.* Again, note preliminarily that $F$ will be finite only on increasing functions so that we need to identify it only on functions $u$ satisfying $u(t+) > u(t-)$ on $S(u)$.

The liminf inequality will be obtained by comparison. Let $\sup_n E_n(u_n) < +\infty$ and $u_n \to u$ in $L^1(0,L)$. Let $v_n(x_i^n) = u_n(x_i^n) - Mx_i^n$. Note that $v_n \to v = u - Mx$, and that

$$E_n(u_n) = \tilde{E}_n(v_n) = \sum_{i=1}^{n} \lambda_n \psi_n\Big(\frac{v_n(x_i^n) - v_n(x_{i-1}^n)}{\lambda_n}\Big),$$

where $\psi_n(z) = \frac{1}{\lambda_n}(J(z+M) - \min J)$. Note that $\psi_n \to +\infty$ if $z \neq 0$, and that

$$\lim_{z\to+\infty} \psi_n(z) = -\frac{\min J}{\lambda_n}.$$

With fixed $k \in \mathbb{N}$ let $\tilde{E}_n^K$ be defined by

$$\tilde{E}_n^K(w) = \sum_{i=1}^{n} \lambda_n \min\Big\{k\Big(\frac{v_n(x_i^n) - v_n(x_{i-1}^n)}{\lambda_n}\Big)^2, \frac{1}{\lambda_n}(\min J - \frac{1}{k})\Big\}.$$

By the results of the previous chapter $\tilde{E}_n^K$ $\Gamma$-converge to $F^K$ defined by

$$F^K(w) = \begin{cases} k\displaystyle\int_{(0,L)} |w'|^2\,dt + (\min J - \frac{1}{k})\#(S(w)) & \text{if } w \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise.} \end{cases}$$

Now, note that for $n$ large enough we have $E_n \geq \tilde{E}_n^K$, so that

$$\liminf_n E_n(u_n) = \liminf_n \tilde{E}_n(v_n) \geq \liminf_n \tilde{E}_n^K(v_n) \geq F^K(v).$$

It will then be sufficient to consider the case $v \in P\text{-}W^{1,2}(0,L)$; that is, $u \in P\text{-}W^{1,2}(0,L)$. In this case we get

$$\liminf_n E_n(u_n) \geq k\int_{(0,L)} |u' - M|^2\,dt + \Big(\min J - \frac{1}{k}\Big)\#(S(u)).$$

By the arbitrariness of $k$ we get the desired inequality.

The limsup inequality is easily obtained. Indeed, if $u \in P\text{-}W^{1,\infty}(0,L)$ is increasing and $u' = M$ a.e. we can take $u_n = u$, in which case $E_n(u_n) = L\min J - \min J\lambda_n + o(\lambda_n)$. □

### Convergence of minimum problems

From the results of the previous section we can easily derive a description of the limiting behaviour of minimizers of minimum problems (127).

**Proposition 2.** *Let $h > ML$; then from every sequence of minimizers of problems (127) we can extract a subsequence converging in $L^1(0,L)$ to an increasing function $u \in P\text{-}W^{1,\infty}(0,L)$ such that $u' = M$ a.e. in $(0,L)$ and, after setting $u(0-) = 0$ and $u(L+) = h$, $u$ has only one jump in $[0,L]$. Moreover we have the estimate $m_n = L\min J - \lambda_n \min J + o(\lambda_n)$ as $n \to +\infty$.*

*Proof.* By the coerciveness conditions on $E_n$, we can suppose that, upon extracting a subsequence, the minimizers of $m_n$ converge in $L^1(0,L)$. We interpret those minimizers also as minimizers of $m_n^{(1)}$. Hence, upon relaxing the boundary conditions, the limit function $u$ solves the problem

$$m^{(1)} = -\min J \,\min\{\#(S(u)) : u \in P\text{-}W^{1,\infty}(0,L),\\ u' = M \text{ a.e.}, u(0-) = 0, u(L+) = h\}.$$

where $S(u)$ is interpreted as a subset of $[0,L]$. The solution of this problem is clearly a function satisfying the thesis of the theorem, and $m^{(1)} = -\min J$. From the convergence of minima $(m_n - L\min J)/\lambda_n = m_n^{(1)} \to m^{(1)} = -\min J$ we complete the proof. □

### Long-range interactions

By taking into account the methods of Sections 1.4 and 1.4 and the proof of Theorem 21 we have the following result.

**Theorem 23.** *Let $K \geq 2$ and let $E_n$ be defined by*

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n J\Big(\frac{u_{i+j} - u_i}{\lambda_n}\Big) \tag{131}$$

*The energies $E_n$ $\Gamma$-converge on $P\text{-}W^{1,1}(0,L)$ with respect to the $L^1(0,L)$ convergence, to the functional $F$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dt & \text{if } u(t+) > u(t-) \text{ on } S(u) \\ +\infty & \text{otherwise} \end{cases} \tag{132}$$

*on $P\text{-}W^{1,1}(0,L)$, where $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is the convex function given by*

$$\psi(z) = \lim_N \min\Big\{\frac{1}{N}\sum_{j=1}^{K}\sum_{i=0}^{N-j} J(u(i+j) - u(i))\\ u : \{0,\dots,N\} \to \mathbb{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N-K\Big\}. \tag{133}$$

*Furthermore, if $K = 2$ then the function $\psi$ is also defined as $\psi = \tilde{J}^{**}$, where*

$$\tilde{J}(z) = J(2z) + \frac{1}{2}\min\{J(z_1) + J(z_2) :\ z_1 + z_2 = 2z\}. \tag{134}$$

*Proof.* The proof follows by using the arguments of Theorems 21, 4 and 5, with $\psi_n^j(z) = J(jz)$, after noting that $\psi$ defined above is convex and bounded at $+\infty$. □

*Remark 14.* The function $\psi$ satisfies the same assumptions as $J$ upon replacing (vi) with $\lim_{z\to+\infty}\psi(z) = C < 0$, but it can be seen that $\tilde{J}$ in general does not satisfy (iv); *i.e.*, is not of convex/concave form.

### 2.4 Renormalization-group approach

A renormalization-group argument suggests a different scaling for Lennard-Jones potentials. In the simplest case of nearest-neighbour interactions, we are led to consider the functionals

$$E_n(u) = \sum_{i=1}^{n} \Big( J\Big(M + \sqrt{\lambda_n}\,\frac{u_i - u_{i-1}}{\lambda_n}\Big) - J(M)\Big).$$

If $J''(M) > 0$ then the arguments for the first- and second-order $\Gamma$-limit above can be used simultaneously, to show the $\Gamma$-convergence to an energy whose domain are piecewise-$H^1$ functions with only increasing jumps, of the form

$$F(u) = \beta \int_0^L |u'|^2\,dt + \alpha\#(S(u)), \qquad u^+ > u^- \text{ on } S(u),$$

where $2\beta = J''(M)$ and $\alpha = -J(M)$. This result can be extended to long-range interactions with a formula for $\alpha$ that takes into account boundary-layer effects on the two sides of the jump (see [24]).

## 3 General convergence results

In order to state and prove general results for the convergence of discrete schemes we will have to describe the $\Gamma$-limits of discrete energies in spaces of functions of bounded variation. We briefly recall some of their properties, referring to [8] for a complete introduction.

### 3.1 Functions of bounded variation

We recall that the space $BV(a,b)$ of *functions of bounded variation* on $(a,b)$ is defined as the space of functions $u \in \mathrm{L}^1(a,b)$ whose *distributional derivative* $Du$ is a signed Borel measure. For each such $u$ there exists $f \in L^1(a,b)$, a (at most countable) set $S(u) \subset (a,b)$, a sequence of real numbers $(a_t)_{t\in S(u)}$ with $\sum_t |a_t| < +\infty$ and a non-atomic measure $D_c u$ singular with respect to the Lebesgue measure such that the equality of measures $Du = f\,\mathcal{L}_1 + \sum_{t\in S(u)} a_t\delta_t + D_c u$ holds. It can be easily seen that for such functions the left hand-side and right hand-side *approximate limits* $u^-(t)$, $u^+(t)$ exist at every point, and that $S(u) = \{t \in \mathbb{R} :\ u^-(t) \neq u^+(t)\}$ and $a_t = u^+(t) - u^-(t) =: [u](t)$. We will write $\dot u = f$, which is an approximate gradient of $u$. $D_c u$ is called the *Cantor part* of $Du$. A sequence $u_j$ converges weakly to $u$ in $BV(a,b)$ if $u_j \to u$ in $\mathrm{L}^1(a,b)$ and $\sup_j |Du_j|(a,b) < +\infty$.

The space $SBV(a,b)$ of *special functions of bounded variation* is defined as the space of functions $u \in BV(a,b)$ such that $D_c u = 0$; *i.e.*, whose distributional derivative $Du$ can be written as $Du = \dot u\,\mathcal{L}_1 + \sum_{t\in S(u)}(u^+(t) - u^-(t))\delta_t$.

This notation describes a particular case of a $SBV$-functions space as introduced by De Giorgi and Ambrosio [30]. We will mainly deal with functionals whose natural domain is that of piecewise-$W^{1,p}$ functions, which is a particular sub-class of $SBV(a,b)$ corresponding to the conditions $\dot{u} \in \mathrm{L}^p(a,b)$ and $\#(S(u)) < +\infty$, but we nevertheless use the more general $SBV$ notation for future reference and for further generalization to higher dimensions (see [7]). For an introduction to $BV$ and $SBV$ functions we refer to the book by Ambrosio, Fusco and Pallara [8], while approximation methods for free-discontinuity problems are discussed by Braides [13].

A class of energies on $SBV(a,b)$ are those of the form

$$\int_{(a,b)} f(\dot{u})\,dt + \sum_{S(u)} g(u^+(t) - u^-(t)),$$

with $f, g : \mathbb{R} \to [0, +\infty]$. Lower semicontinuity conditions on $\mathcal{F}$ imply that $f$ is lower semicontinuous and convex and $g$ is lower semicontinuous and subadditive; *i.e.*, $g(x+y) \le g(x)+g(y)$. The latter can be interpreted as a condition penalizing fracture fragmentation, whereas convexity penalizes oscillations. If $g$ is not lower semicontinuous and subadditive then we may consider its *lower semicontinuous and subadditive envelope*; *i.e.*, the greatest lower semicontinuous and subadditive function not greater than $g$, that we denote by $\mathrm{sub}^- g$. For a discussion on the role of this condition for the lower semicontinuity of $\mathcal{F}$ we refer to [13] Section 2.2 or [14]. Energies in $BV$ must satisfy further compatibility conditions between $f$ and $g$ (see *e.g.* Theorem 24 below and the subsequent remark).

The following theorem is an easy corollary of [5] Theorem 6.3 and will be widely used in the next section.

**Theorem 24.** *For all $n \in \mathbb{N}$ let $f_n, g_n : \mathbb{R} \to [0, +\infty]$ be lower semicontinuous functions. Let $\alpha > 0$ exists such that*

(1) *$f_n$ is convex and*

$$\alpha(|z| - 1) \le f_n(z) \qquad \textit{for every } z \in \mathbb{R},$$

(2) *$g_n$ is subadditive and*

$$\alpha(|z| - 1) \le g_n(z) \qquad \textit{for every } z \in \mathbb{R}.$$

*and suppose that $f, g : \mathbb{R} \to [0, +\infty]$ exist such that $\Gamma\text{-}\lim_n f_n = f$ on $\mathbb{R}$ and $\Gamma\text{-}\lim_n g_n = g$ on $\mathbb{R} \setminus \{0\}$. For notation's convenience we set $g(0) = 0$. Let $\mathcal{H}_n : BV(a,b) \to [0, +\infty]$ be defined as*

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\int_a^b f_n(\dot{u})\,dx + \sum_{S(u)} g_n([u]) & \textit{if } u \in SBV(a,b) \\ +\infty & \textit{otherwise.} \end{cases}$$

*Then $\mathcal{H}_n$ $\Gamma$-converge with respect to the weak topology of $BV(a,b)$ to the functional $\mathcal{H} : BV(a,b) \to [0,+\infty)$ defined by*

$$\mathcal{H}(u) := \int_0^l \overline{f}(\dot{u})\,dx + \sigma^+ D_c u^+(a,b) + \sigma^- D_c u^-(a,b) + \sum_{S(u)} \overline{g}([u])$$

*(recall that $D_c u^\pm$ denote the positive/negative part of the Cantor measure $D_c u$), where*

$$\overline{f}(z) := \inf\{f(z_1) + g^0(z_2) : z = z_1 + z_2\},$$
$$\overline{g}(z) := \inf\{f^\infty(z_1) + g(z_2) : z = z_1 + z_2\},$$

$$f^\infty(z) = \lim_{t\to+\infty} \frac{f(tz)}{t}, \quad g^0(z) = \lim_{t\to+\infty} tg\Big(\frac{z}{t}\Big), \quad \textit{and} \quad \sigma^\pm = \lim_{t\to+\infty} \frac{\overline{f}(\pm t)}{t}$$

*for all $z \in \mathbb{R}$.*

*Remark 15.* Note that if we take $g_n = g$ and $f_n = f$ we recover the well-known compatibility hypothesis $f^\infty = g^0$ for weakly lower semicontinuous functionals on $BV(a,b)$.

If $f(0) = 0$ then it can be easily seen that $\overline{f} = (f \wedge g^0)^{**}$ and $\overline{g} = \mathrm{sub}^-(f^\infty \wedge g)$.

### 3.2 Nearest-neighbour interactions

For future reference, we state and prove the convergence results allowing for a more general dependence on the underlying lattice than in the previous chapters, at the expense of a slightly more complex notation.

We begin by identifying the functions defined on a lattice with a subset of measurable functions. Consider an open interval $(a,b)$ of $\mathbb{R}$ and two sequences $(\lambda_n)$, $(a_n)$ of positive real numbers with $a_n \in [a, a+\lambda_n)$ and $\lambda_n \to 0$. For $n \in \mathbb{N}$ let $a \le x_n^1 < \ldots < x_n^{N_n} < b$ be the partition of $(a,b)$ induced by the intersection of $(a,b)$ with the set $a_n + \lambda_n\mathbb{Z}$. We define $\mathcal{A}_n(a,b)$ the set of the restrictions to $(a,b)$ of functions constant on each $[a+k\lambda_n, a+(k+1)\lambda_n)$, $k \in \mathbb{Z}$. A function $u \in \mathcal{A}_n(a,b)$ will be identified by $N_n + 1$ real numbers $c_n^0, \ldots, c_n^{N_n}$ such that

$$u(x) = \begin{cases} c_n^i & \text{if } x \in [x_n^i, x_n^{i+1}),\ i = 1,\ldots,N_n - 1 \\ c_n^0 & \text{if } x \in (a, x_n^1) \\ c_n^{N_n} & \text{if } x \in [x_n^{N_n}, b). \end{cases} \tag{135}$$

For $n \in \mathbb{N}$ let $\psi_n : \mathbb{R} \to [0,+\infty]$ be a given Borel function and define $E_n : L^1(a,b) \to [0,+\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{i=1}^{N_n-1} \lambda_n \psi_n\left(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n}\right) & x \in \mathcal{A}_n(a,b) \\ +\infty & \text{otherwise in } L^1(a,b). \end{cases} \tag{136}$$

The following sections contain the description of the asymptotic behaviour of $E_n$ as $n \to +\infty$.

## Potentials with local superlinear growth

We first treat the case when the potentials $\psi$ satisfy locally a growth condition of order $p > 1$. This is the case of non-convex potentials introduced by Blake and Zisserman and of the scaled Lennard-Jones potentials which justify Griffith theory of fracture as a first-order effect.

**Theorem 25.** *For all $n \in \mathbb{N}$ let $T_n^{\pm} \in \mathbb{R}$ exist with*

$$\lim_n T_n^{\pm} = \pm\infty, \qquad \lim_n \lambda_n T_n^{\pm} = 0, \tag{137}$$

*and such that, if we define $f_n, g_n : \mathbb{R} \to [0, +\infty]$ as*

$$f_n(z) = \begin{cases} \psi_n(z) & \text{if } T_n^- \le z \le T_n^+ \\ +\infty & \text{if } z \in \mathbb{R} \setminus [T_n^-, T_n^+] \end{cases} \tag{138}$$

$$g_n(z) = \begin{cases} \lambda_n \psi_n\left(\dfrac{z}{\lambda_n}\right) & z \in \mathbb{R} \setminus [\lambda_n T_n^-, \lambda_n T_n^+] \\ +\infty & \text{otherwise} \end{cases} \tag{139}$$

*the following conditions are satisfied: there exists $p > 1$ such that*

$$f_n(z) \ge |z|^p \qquad \forall z \in \mathbb{R} \tag{140}$$

$$g_n(z) \ge c > 0 \qquad \forall z \ne 0 \tag{141}$$

*and, moreover, there exist $f, g : \mathbb{R} \to [0, +\infty]$, such that*

$$\Gamma\text{-}\lim_n f_n^{**} = f \ \text{ on } \mathbb{R}, \tag{142}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- g_n = g \ \text{ on } \mathbb{R}. \tag{143}$$

*Then, $(E_n)_n$ $\Gamma$-converges to $F$ with respect to the convergence in measure on $L^1(a,b)$, where*

$$F(u) = \begin{cases} \displaystyle\int_a^b f(\dot u)\, dt \ + \sum_{t \in S(u)} g([u](t)) & u \in SBV(a,b) \\ +\infty & \text{otherwise in } \ L^1(a,b). \end{cases}$$

*Remark 16.* Note that hypotheses (142) and (143) are not restrictive upon passing to a subsequence by a compactness argument. This remark also holds for Theorems 26 and 27. Moreover, if $f$ is finite everywhere then $\Gamma$-convergence in (142) can be replaced by pointwise convergence.

*Proof.* For simplicity of notation we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case following by simple modifications. Without loss of generality we may assume

$$\sup_n \inf_{z\in\mathbb{R}} f_n(z) < +\infty; \tag{144}$$

otherwise we trivially have $f \equiv +\infty$ and consequently $F \equiv +\infty$.

With fixed $u \in L^1(a,b)$ and a sequence $(u_n) \subseteq \mathcal{A}_n(a,b)$ such that $u_n \to u$ in measure and $\sup_n E_n(u_n) < +\infty$. Up to a subsequence, we can suppose in addition that $u_n$ converges to $u$ pointwise a.e. We now construct for each $n \in \mathbb{N}$ a function $v_n \in SBV(a,b)$ and a free-discontinuity energy such that $v_n$ still converges to $u$ and we can use that energy to give a lower estimate for $E_n(u_n)$. Set

$$I_n := \left\{ i \in \{1,\dots,N_n-1\} : \left| \frac{u_n(x_n^{i+1}) - u_n(x_n^i)}{\lambda_n} \right| > T_n \right\} \tag{145}$$

and

$$v_n(x) := \begin{cases} u_n(x_n^1) & \text{if } x \in (a, x_n^1) \\ c_n^i + \dfrac{(c_n^{i+1} - c_n^i)}{\lambda_n}(x - x_n^i) & x \in [x_n^i, x_n^{i+1}),\ i \notin I_n \\ u_n(x) & x \text{ elsewhere in } (a,b). \end{cases} \tag{146}$$

We have that, for $\varepsilon > 0$ fixed,

$$\begin{aligned} &\{x : |v_n(x) - u_n(x)| > \varepsilon\} \\ &\subseteq \{x \in [x_n^i, x_n^{i+1}),\ i \notin I_n,\ |u_n(x_n^{i+1}) - u_n(x_n^i)| > \varepsilon\} \cup (a, x_n^1). \end{aligned} \tag{147}$$

Since, for $i \notin I_n$ we have $|u_n(x_n^{i+1}) - u_n(x_n^i)| \le \lambda_n T_n$, then $\{x : |v_n(x) - u_n(x)| > \varepsilon\}$ consists at most of the interval $(a, x_n^1)$ if $n$ is large enough. Hence, the sequence $(v_n)_n$ converges to $u$ in measure and pointwise a.e. Moreover, by (141)

$$c\#I_n \le E_n(u_n) \le M, \tag{148}$$

with $M = \sup_n E_n(u_n)$. By the equiboundedness of $\#I_n$, we can suppose that $S(v_n) = \{x_n^{i+1}\}_{i\in I_n}$ tends to a finite set. For the local nature of the arguments in the following reasoning, we can also assume that $S$ consists of only one point $x_0 \in (a,b)$.

Now, consider the sequence $(w_n)_n$ defined by

$$w_n(x) = \begin{cases} v_n(a) + \displaystyle\int_{(a,x)} \dot v_n(t)\,dt & \text{if } x < x_0 \\ v_n(a) + \displaystyle\int_{(a,x)} \dot v_n(t)\,dt + \sum_{t\in S(v_n)} [v_n](t) & \text{if } x \ge x_0. \end{cases} \tag{149}$$

Note that $w_n(a) = v_n(a)$, $\dot w_n = \dot v_n$, $S(w_n) = \{x_0\}$ and $[w_n](x_0) = \sum_{t\in S(v_n)}[v_n](t)$. Such a sequence still converges to $u$ a.e. Indeed, since $x_0$ is the limit point of the sets $S(v_n)$, for any $\eta > 0$ fixed we can find $n_0(\eta) \in \mathbb{N}$ such that for any $n \ge n_0(\eta)$ and for any $i \in I_n$ $|x_0 - x_n^{i+1}| < \eta$. Hence, by construction, for any $n \geq n_0(\eta)$ and for any $x \in (a,b) \setminus [x_0 - \eta, x_0 + \eta]$, $w_n(x) = v_n(x)$, that is, the two sequences $(v_n)$ and $(w_n)$ have the same pointwise limit. Since $\dot{w}_n = \dot{v}_n$ on $(a,b)$, by (140) we have that $\|\dot{w}_n\|_{L^p(a,b)} \leq M$. Then, using Poincaré's inequality on each interval, it can be easily seen that $(w_n)_n$ is equibounded in $W^{1,p}((a,b)\setminus\{x_0\})$. Since it also converges to $u$ pointwise a.e., by using a compactness argument, we get that $u \in W^{1,p}((a,b)\setminus\{x_0\})$ and, up to subsequences,

$$\dot{w}_n \rightharpoonup \dot{u} \text{ weakly in } L^p(a,b).$$

Moreover, since for any two points $a < x_1 < x_0 < x_2 < b$ we have

$$w_n(x_2) = w_n(x_1) + \int_{x_1}^{x_2} \dot{w}_n \, dt + [w_n](x_0)$$

$$u(x_2) = u(x_1) + \int_{x_1}^{x_2} \dot{u} \, dt + [u](x_0),$$

taking points $x_1, x_2$ in which $w_n$ converges to $u$ and passing to the limit as $n \to +\infty$, we have

$$[w_n](x_0) \to [u](x_0). \tag{150}$$

We can now rewrite our functionals in terms of $v_n$:

$$\begin{aligned} E_n(u_n) &= \sum_{i \notin I_n} \lambda_n \psi_n(\dot{v}_n) + \sum_{i \in I_n} g_n([v_n](x_n^{i+1})) \\ &= \int_a^b f_n(\dot{v}_n) \, dt + \sum_{t \in S(v_n)} g_n([v_n](t)). \end{aligned}$$

From (148) we also have

$$\begin{aligned} E_n(u_n) &\geq \int_a^b f_n(\dot{v}_n) \, dt + \mathrm{sub}^- g_n\Big( \sum_{t \in S(v_n)} [v_n](t)\Big) \\ &\geq \int_a^b f_n^{**}(\dot{w}_n) \, dt + \mathrm{sub}^- g_n([w_n](x_0)) \end{aligned}$$

as $n \to +\infty$. Passing to the liminf as $n \to +\infty$, using (150) we have

$$\begin{aligned} \liminf_n E_n(u_n) &\geq \liminf_n \int_a^b f_n^{**}(\dot{w}_n) \, dt + \liminf_n \mathrm{sub}^- g_n([w_n](x_0)) \\ &\geq \int_a^b f(\dot{u}) \, dt + g([u](x_0)) \end{aligned}$$

as desired.

We now turn our attention to the construction of recovery sequences for the $\Gamma$-limsup. We may assume in what follows that $\inf_{z \in \mathbb{R}} f_n(z) = f_n(0)$.

*Step* 1 We first prove the limsup inequality for $u$ affine on $(a,b)$. Set $\xi=\dot u$; we can assume, upon a slight translation argument, that $f(\xi)=\lim_n f_n^{**}(\xi)$. Then, for each $n$ in $\mathbb{N}$ we can find $\xi_n^1,\xi_n^2\in\mathbb{R}$, $t_n\in[0,1]$ such that

$$\begin{aligned} |t_n\xi_n^1+(1-t_n)\xi_n^2-\xi| &\le \frac{\sqrt{\lambda_n}}{2(b-a)}\\ t_nf_n(\xi_n^1)+(1-t_n)f_n(\xi_n^2) &\le f_n^{**}(\xi)+o(1)\\ |\xi_n^i| &\le c=c(\xi). \end{aligned} \tag{151}$$

Note that in the last inequality the choice of the constant $c$ can be chosen independent of $n$ thanks to (140) and (144). It can be easily seen that it is not restrictive to make the following assumptions on $\xi_n^i$:

$$\xi_n^1>\xi,\quad f_n(\xi_n^1)\le f_n(\xi_n^2),\quad (|\xi_n^1|+|\xi_n^2|)\sqrt{\lambda_n}\le 1. \tag{152}$$

We define a piecewise-affine function $v_n\in\ L^1(a,b)$ with the following properties:

$$v_n(x)=u(x)\text{ on }(a,x_n^1],\qquad \dot v_{n\,|\,[x_n^i,x_n^{i+1})}:=v_n^i\in\{\xi_n^1,\xi_n^2\},$$

and $v_n^i$ is defined recursively by

$$\begin{cases} v_n^1=\xi_n^1\\ v_n^{i+1}=\begin{cases} v_n^i & \text{if } \frac{\sqrt{\lambda_n}}{2}\le v_n(a_n)+\sum\limits_{j=1}^{i}v_n^j\lambda_n+v_n^i\lambda_n-u(x_n^{i+1})\le\sqrt{\lambda_n}\\ \xi_n^1+\xi_n^2-v_n^i & \text{otherwise.}\end{cases}\end{cases} \tag{153}$$

Since $0\le v_n-u\le\sqrt{\lambda_n}$ by definition, $(v_n)_n$ converges to $u$ uniformly, and hence in measure and, moreover,

$$\beta_n^1:=\#\left\{i\in\{0,\dots,N_n\}:\ v_n^i=\xi_n^1\right\}\ge t_nN_n. \tag{154}$$

Indeed, from (151), (152) and (153) we deduce

$$\begin{aligned}\lambda_nN_n\left(t_n(\xi_n^1-\xi)+(1-t_n)(\xi_n^2-\xi)\right)\le\frac{\sqrt{\lambda_n}}{2}&\le v_n(x_n^{N_n})-u(x_n^{N_n})\\ &=\beta_n^1(\xi_n^1-\xi)\lambda_n+(N_n-\beta_n^1)(\xi_n^2-\xi)\lambda_n,\end{aligned}$$

so that

$$(\beta_n^1-t_nN_n)(\xi_n^1-\xi_n^2)\ge 0.$$

Now, consider the sequence $(u_n)\subseteq A_n(a,b)$ defined by

$$u_n(x_n^i)=v_n(x_n^i)\text{ for }i=1,\dots,N_n,\qquad u_n(a)=v_n(a)\text{ and }u_n(b)=v_n(b).$$

Since (147) still holds with $u_n,v_n$ as above, it can be easily checked that $(u_n)_n$ converges to $u$ in measure. Hence, recalling (151), (152) and (154),

$$\begin{aligned} E_n(u_n) &= \lambda_n f_n(\xi_n^1)\beta_n^1 + (N_n - \beta_n^1)\lambda_n f_n(\xi_n^2) \\ &\le t_n \lambda_n N_n f_n(\xi_n^1) + (1-t_n) N_n \lambda_n f_n(\xi_n^2) \\ &\le N_n \lambda_n \left(f_n^{**}(\xi) + o(1)\right) \le (b-a) f_n^{**}(\xi) + o(1). \end{aligned}$$

Taking the limsup as $n \to +\infty$ we get

$$\limsup_n E_n(u_n) \le f(\xi)(b-a) = F(u).$$

The same construction as above works also in the case of a piecewise-affine function: let $[a,b] = \bigcup [a_j, b_j]$ with $a_1 = a$, $b_j = a_{j+1}$ and $\dot u$ constant on each $(a_j, b_j)$, then it suffices to repeat the procedure above on each $(a_j, b_j)$ to provide functions $v_n^j$ in $A_n(a_j, b_j)$ such that

$$\begin{aligned} & v_n^j \to u_{|\,(a_j,b_j)} \qquad \text{in measure} \\ & \limsup_n \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left( \frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) \le \int_{a_j}^{b_j} f(\dot u)\, dx. \end{aligned}$$

With $j$ fixed define $y_n^j := \max\{x_n^i \in (a_j, b_j)\}$. Then, the recovery sequence $u_n$ is defined in $(a_j, b_j)$ as

$$u_n(x) = v_n^j(x) - \sum_{\ell < j} (v_n^{\ell+1}(y_n^\ell + \lambda_n) - v_n^\ell(y_n^\ell)).$$

Since $u(x) + \frac{\sqrt{\lambda_n}}{2} \le v_n^j(x) \le u(x) + \sqrt{\lambda_n}$ by construction, and $|u(y_n^\ell + \lambda_n) - u(y_n^\ell)| \le c\lambda_n$, we have that $u_n \to u$ in measure and

$$E_n(u_n) = \sum_j \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left( \frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) + c f_n(0) \lambda_n.$$

By a density argument we can extend the result to functions in $W^{1,p}(a,b)$.

*Step* 2 Let $u$ be of the form $z\chi_{(x_0,b)}$ with $g(z) < +\infty$ and let $z_n$ be a recovery sequence for $g(z) = \Gamma\text{-}\lim_n \mathrm{sub}^- g_n(u)$. The sequence $\mathrm{sub}^- g_n(z_n)$ is bounded, hence, by (141), upon possibly considering a suitable subsequence, there exists an integer $N$ not depending on $n$ such that

$$\mathrm{sub}^- g_n(z_n) = \sup_\varepsilon \inf \left\{ \sum_{i=1}^N g_n(z^i) : \left| \sum_{i=1}^N z^i - z_n \right| < \varepsilon \right\}.$$

Hence, for all $n$ we can find $N$ points $\{z_n^1, \dots, z^N\}$ such that

$$\lim_n \sum_{i=1}^N z_n^i = z \quad \text{and} \quad \lim_n \sum_{i=1}^N g_n(z_n^i) = g(z). \tag{155}$$

Let $i_n \in \{1,\dots,N_n\}$ be the index such that $x_0 \in [x_n^{i_n}, x_n^{i_n+1})$ and, for $n$ large, define $w_n$ as in (135) with

$$c_n^i = \begin{cases} 0 & \text{if } i \le i_n \\ \sum\limits_{j\le(i-i_n)} \left(z_n^j\right) & \text{if } i_n < i \le i_n + N \\ \sum\limits_{j=1}^{N} \left(z_n^j\right) & \text{if } i > i_n + N. \end{cases} \tag{156}$$

Clearly $(w_n)_n \to u$ in measure and

$$E_n(w_n) = \sum_{i=1}^{N} \lambda_n \psi_n\left(\frac{z_n^i}{\lambda_n}\right) = \sum_{i=1}^{N} g_n(z_n^i) + (b-a) f_n(0)\,;$$

the estimate follows from (155) by passing to the limit as $n \to +\infty$.

*Step* 3 Let $u \in SBV(a,b)$ be such that $F(u) < +\infty$, then

$$u = v + w \text{ with } v(x) = \int_a^x \dot{u}\,dt + c \text{ and } w(x) = \sum_{j=1}^{m} z_j \chi_{[x_j,b)}.$$

For $j = 1,\dots m$ let $w_n^j$ be defined as in Step 2 with jumps in $\bigcup_j \{x_n^{i+i_{n,j}}\}_{i=1}^{N_j}$ and let $v_n$ be a recovery sequence for $v$ such that it is constant on each $[x_n^{i_{n,j}}, x_n^{i_{n,j}} + \lambda_n N_j)$. The sequence $u_n = v_n + \sum\limits_{j=1}^{m} w_n^j$ converges in measure to $u$ and

$$\limsup_n E_n(u_n) = \limsup_n \Big(E_n(v_n) + \sum_{j+1}^{m} E_n(w_n^j)\Big) \le F(v) + F(w) = F(u),$$

as desired. □

**Corollary 2.** *Let $\psi_n : \mathbb{R} \to [0,+\infty]$ satisfy the hypotheses of Theorem* 25*. Assume that, in addition, for all $n \in \mathbb{N}$, $f_n = f_n^{**}$ on $[T_n^-, T_n^+]$ and $g_n = \mathrm{sub}^- g_n$ on $\mathbb{R} \setminus [\lambda_n T_n^-, \lambda_n T_n^+]$. Then, for any $u \in L^1(a,b)$, $E_n(u)$ $\Gamma$-converges to $F(u)$ with respect to the strong topology of $L^1(a,b)$.*

*Proof.* It suffices to produce a recovery sequence converging strongly in $L^1(a,b)$. Note that in Step 1, by the convexity of $f_n$, we can choose $\xi_n^1 = \xi_n^2 = \xi_n$ in (151). Then $v_n = u$ and $u_n$ turns out to be the piecewise-constant interpolation of $u$ at points $\{x_n^i\}$. It is easy to check that $u_n \to u$ strongly in $L^1(a,b)$. It remains to show that also for functions of the form $z\chi_{[x_0,b)}$ it is possible to exhibit a sequence that converges strongly in $L^1(a,b)$. To this end it suffices to note that in Step 2, since $g_n = \mathrm{sub}^- g_n$ locally on $\mathbb{R} \setminus \{0\}$, we can find a sequence $(z_n)$ such that (155) is replaced by $\lim_n z_n = z$ and $\lim_n g_n(z_n) = g(z)$. Hence, the sequence $w_n$ defined by (156) converges to $u$ strongly in $L^1(a,b)$ and it is a recovery sequence. □

*Remark 17.* Note that the hypotheses of the previous corollary are satisfied if $\psi_n$ is convex and lower semicontinuous on $[T_n^-, T_n^+]$ and concave and lower semicontinuous on $(-\infty, T_n^-]$ and $[T_n^+, +\infty)$

## Potentials with linear growth

In this section we will consider energy potentials $\psi_n$ such that

$$\psi_n(z) \geq \alpha(|z| - 1) \qquad \forall z \in \mathbb{R} \tag{157}$$

for some $\alpha > 0$. For this kind of energies we can still prove a convergence result to a free-discontinuity energy, whose volume and surface densities are obtained by a suitable interaction of the limit functions $f$, $g$ of the two 'regularized' scalings of $\psi_n$. Note that in the following statement the sequences $T_n^\pm$ are arbitrary.

**Theorem 26.** *Let $\psi_n : \mathbb{R} \to [0, +\infty]$ satisfy (157). For all $n \in \mathbb{N}$ let $T_n^\pm \in \mathbb{R}$ satisfy properties (137) and let $f_n, g_n : \mathbb{R} \to [0, +\infty]$ be defined as in (138). Assume that $f, g : \mathbb{R} \to [0, +\infty]$ exist such that*

$$\Gamma\text{-}\lim_n f_n^{**} = f \ \text{ on } \mathbb{R}, \tag{158}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- g_n = g \ \text{ on } \mathbb{R} \setminus \{0\}. \tag{159}$$

*For notation's convenience we set $g(0) = 0$. Then, $(E_n)_n$ $\Gamma$-converges to $F$ with respect to the convergence in $L^1(a, b)$ and the convergence in measure, where*

$$F(u) = \begin{cases} \displaystyle\int_a^b \overline{f}(\dot{u})\, dx + \sum_{S(u)} \overline{g}([u]) + c_1 Du_c^+(a, b) + c_{-1} Du_c^-(a, b) & \text{if } u \in BV(a, b) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\overline{f}(z) := \inf\{f(z_1) + g^0(z_2) : z_1 + z_2 = z\},$$
$$\overline{g}(z) := \inf\{f^\infty(z_1) + g(z_2) : z_1 + z_2 = z\},$$

$c_1 := \overline{f}^\infty(1)$ *and* $c_{-1} := \overline{f}^\infty(-1)$.

*Remark 18.* Thanks to (157) the theorem can be restated also with respect to the weak convergence in $BV(a, b)$. Indeed, sequences converging in measure along which the functionals $E_n$ are equibounded are weakly compact in $BV$.

*Proof.* Again we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case being achieved by slight modifications. Let $u_n, u \in L^1(a, b)$ be such that $u_n \to u$ in measure and $E_n(u_n) \leq c$. Analogously to the proof of Theorem 25, we will estimate $E_n(u_n)$ by a free-discontinuity energy computed on a sequence $v_n$

converging to $u$ weakly in $BV(a,b)$. Let $I_n$ and $v_n$ be defined as in (145) and (146), respectively. Note that $v_n \to u$ in measure and that $v_n$ has equibounded total variation on $(a,b)$. Indeed, by hypothesis (157) we have

$$|Dv_n|(a,b) = \sum_{i=1}^{N_n-1} |u_n(x_n^{i+1}) - u_n(x_n^i)| \le \frac{1}{\alpha} E_n(u_n) + c.$$

From this inequality we easily get that $u \in BV(a,b)$, in particular the $\Gamma$-liminf is finite only on $BV(a,b)$.

Up to passing to a subsequence we may assume that $v_n$ converges to $u$ weakly in $BV(a,b)$; moreover, by construction we have

$$E_n(u_n) \ge \int_a^b f_n^{**}(\dot v_n)\,dt + \sum_{S(v_n)} \mathrm{sub}^- g_n([v_n]).$$

Hence, it suffices to apply Theorem 24 to the functionals on the left hand side to get the $\Gamma$-liminf inequality.

To obtain the converse inequality it suffices to provide a sequence $v_n$ converging to $u$ in $L^1(a,b)$ such that

$$\limsup_n E_n(v_n) \le F_1(u) := \int_a^b f(\dot u)\,dt + \sum_{S(u)} g([u])$$

when $u \in SBV(a,b)$. The general estimate will be then obtained by relaxation (*i.e.* by taking $f_n = f$ and $g_n = g$ in Theorem 24). By a standard approximation argument it is sufficient to prove this inequality in the simpler cases of $u$ linear and of $u$ with a single jump. Let $u(t) = \xi t$; we may assume that $f(\xi) < +\infty$. Moreover, we may assume in what follows that $\inf_{z\in\mathbb{R}} f_n(z) = f_n(0)$. Then we can find $\xi_n^1, \xi_n^2$ such that the analogue of (151) holds. In this case, $|\xi_n^1|, |\xi_n^2|$ are not necessarily equibounded; nevertheless we have by definition $|\xi_n^1|, |\xi_n^2| \le T_n$ since $f(\xi) < +\infty$. Thus we can construct the functions $v_n$ as in the proof of the $\Gamma$-limsup-inequality of Theorem 25, up to a slight modification. Indeed, if we replace $\sqrt{\lambda_n}$ with $\lambda_n T_n$ in (151) and (153), all those inequalities still hold. In particular we have that $|u(x) - v_n(x)| \le \lambda_n T_n$ in $(a,b)$. Thus $v_n$ is a recovery sequence converging to $u$ in $L^\infty(a,b)$.

As for the case of $u = z\chi_{(x_0,b)}$ with $g(z) < +\infty$, let $z_n = \sum_{i=1}^{M_n} z_n^i$ be such that $g(z) = \lim_n \sum_{i=1}^{M_n} g_n(z_n^i)$. Note that since $\lim_n \sum_{i=1}^{M_n} z_n^i = z$, by taking (157) into account, we may assume that $\sum_{i=1}^{M_n} (z_n^i)^+ \le c|z|$ and the same for the negative terms. We may assume also that $|z_n^i| \ge \lambda_n T_n$ for any $i$. Hence, by arguing separately on the positive and negative part of $(z_n^i)$, we easily get that

$$M_n \le \frac{c|z|}{\lambda_n T_n}. \tag{160}$$

Finally we can construct a sequence of functions $w_n$ defined as in (156) where we replace $N$ with $M_n$. By taking (160) into account we easily get

$$|\{x : w_n(x) \neq u(x)\}| \leq \lambda_n M_n \leq \frac{c|z|}{T_n}.$$

Then $w_n \to u$ in $L^\infty(a,b)$ and, by construction,

$$\limsup_n E_n(w_n) = g(z) = F_1(u).$$

The desired upper estimate follows then by standard arguments. □

### Potentials of Lennard-Jones type

We now treat the case of potentials with non-symmetric growth conditions, which still ensure weak-$BV$ compactness of sequences with equibounded energies. These conditions are satisfied for example by Lennard-Jones potentials.

**Theorem 27.** *Let $\psi_n : \mathbb{R} \to [0, +\infty]$ satisfy*

$$\psi_n(z) \geq (|z|^p - 1) \qquad \text{for all } z < 0. \tag{161}$$

*for some $p > 1$. For all $n \in \mathbb{N}$ let $T_n \in \mathbb{R}$ satisfy*

$$\lim_n T_n = +\infty, \qquad \lim_n \lambda_n T_n = 0, \tag{162}$$

*and let $f_n, g_n : \mathbb{R} \to [0, +\infty]$ be defined by*

$$f_n(z) = \begin{cases} \psi_n(z) & z \leq T_n \\ +\infty & z > T_n^+ \end{cases} \tag{163}$$

$$g_n(z) = \begin{cases} \lambda_n \psi_n\left(\dfrac{z}{\lambda_n}\right) & \text{if } z > \lambda_n T_n \\ +\infty & \text{otherwise.} \end{cases} \tag{164}$$

*Assume that there exist $f, g : \mathbb{R} \to [0, +\infty]$ such that*

$$\Gamma\text{-}\lim_n f_n^{**} = f \ \text{ on } \mathbb{R}, \tag{165}$$

$$\Gamma\text{-}\lim_n \text{sub}^- g_n = g \ \text{ on } \mathbb{R} \setminus \{0\}. \tag{166}$$

*For notation's convenience we set $g(0) = 0$. Then, $(E_n)_n$ $\Gamma$-converges to $F$ with respect to the convergence in $L^1_{\text{loc}}(a,b)$ and the convergence in measure, where*

$$F(u) = \begin{cases} \displaystyle\int_a^b \overline{f}(\dot{u})\,dx + \sum_{S(u)} \overline{g}([u]) + \sigma D u_c^+(a,b) & \text{if } u \in BV_{\text{loc}}(a,b)\ D_c u^- = 0 \\ & \text{and } [u] > 0 \text{ on } S(u) \\ +\infty & \text{otherwise in } L^1(a,b). \end{cases}$$

*where $\overline{f}$ and $\overline{g}$ are defined as in Theorem 26 and $\sigma := \overline{f}^\infty(1)$.*

*Proof.* Let $u_n \to u$ in measure and be such that $E_n(u_n) \le c$, and assume that $u_n \to u$ also pointwise. Set

$$I_n = \{i \in \{1, \dots, N_n\} : \ u_n(x_n^{i+1}) - u_n(x_n^i) > \lambda_n T_n\},$$

and let $v_n$ be the sequence of functions defined as in (146) with this choice of $I_n$. Note that $v_n \to u$ in measure. By taking hypothesis (161) into account we have the following estimate on the negative part of the (classical) derivative of $v_n$

$$\int_a^b |(\dot v_n)^-|^p \, dt \le \sum_{i \notin I_n} \lambda_n \Big( \frac{(u_n(x_n^{i+1}) - u_n(x_n^i))^-}{\lambda_n} \Big)^p \le \frac{E_n(u_n)}{\alpha} + c.$$

Hence, with fixed $\delta > 0$ and with fixed $x_1, x_2$ points in $(a, a+\delta)$, $(b-\delta, b)$, respectively, in which $v_n$ converges pointwise to $u$, we get

$$\Big| \int_{a+\delta}^{b-\delta} (\dot v_n)^+ \, dt + \sum_{S(v_n) \cap (a+\delta, b-\delta)} [v_n] \Big| \le |v_n(x_2) - v_n(x_1)| + \int_a^b (\dot v_n)^- \, dt.$$

It follows that $v_n$ is bounded in $BV_{\rm loc}(a,b)$. Since $v_n \to u$ in measure we get that $v_n$ converges in $BV_{\rm loc}(a,b)$ to $u$ and hence $u \in BV_{\rm loc}(a,b)$. With fixed $\eta > 0$, consider

$$f_n^\eta(z) = (f_n)^{**}(z) + \eta|z|, \qquad g_n^\eta(z) = {\rm sub}^- g_n(z) + \eta |z|.$$

For every $\delta \in (0, (b-a)/2)$ we have

$$E_n(u_n) + \eta c(\delta) \ge \int_{a+\delta}^{b-\delta} f_n^\eta(\dot v_n) \, dt + \sum_{S(v_n) \cap (a+\delta, b-\delta)} g_n^\eta([v_n]),$$

where $c(\delta) = \sup_n |Dv_n|(a+\delta, b-\delta)$. We can apply Theorem 24 and obtain for every $\eta$ and $\delta$

$$\begin{aligned} &\liminf_n E_n(u_n) + \eta c(\delta) \\ &\ge \int_{a+\delta}^{b-\delta} \overline{f}(\dot u) \, dt + \overline{f}^\infty(1) |D_c u^+|(a+\delta, b-\delta) + \sum_{S(u) \cap (a+\delta, b-\delta)} \overline{g}([u]). \end{aligned}$$

By letting $\eta \to 0$ and subsequently $\delta \to 0$ we obtain the desired inequality.

The construction of a recovery sequence for the $\Gamma$-limsup follows the same procedure as in the proof of Theorem 26. □

### Examples

*Example 4.* (i) (Blake-Zisserman model) The typical example of a sequence of functions which satisfy the hypotheses of Theorem 25 (and indeed of Corollary 2) is given (fixed $(\lambda_n)$ converging to 0 and $C > 0$) by

$$\psi_n(z) = \frac{1}{\lambda_n}\big((\lambda_n z^2) \wedge C\big),$$

with $p = 2$, $T_n = \sqrt{C/\lambda_n}$,

$$f_n(z) = \begin{cases} z^2 & |z| \le \sqrt{C/\lambda_n} \\ +\infty & \text{otherwise,} \end{cases} \qquad g_n(z) = \begin{cases} C & |z| > \sqrt{C/\lambda_n} \\ +\infty & |z| \le \sqrt{C/\lambda_n}, \end{cases}$$

so that

$$F(u) = \int_a^b |\dot u|^2\,dt + C\#(S(u))$$

on $SBV(a,b)$

(ii) Theorem 25 allows also to treat asymmetric cases. As an example, let

$$\psi_n(z) = \begin{cases} \frac{1}{\lambda_n}\big((\lambda_n z^2) \wedge C\big) & \text{if } z > 0 \\ z^2 & \text{if } z \le 0. \end{cases}$$

In this case the $\Gamma$-limit (with respect to both the convergence in measure and $L^1$ convergence) is given by

$$F(u) = \begin{cases} \displaystyle\int_a^b |\dot u|^2\,dt + C\#(S(u)) & \text{if } u \in SBV(a,b) \text{ and } u^+ > u^- \text{ on } S(u) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that

$$g(z) = \begin{cases} C & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ +\infty & \text{if } z < 0 \end{cases}$$

forbids negative jumps.

(iii) (Lennard-Jones type potentials) Let $\psi : \mathbb{R} \to [0, +\infty]$ be a lower semicontinuous function and satisfy $\psi(z) = 0$ if and only if $z = 0$, $\psi(z) \ge \alpha(|z|^p - 1)$ for $z < 0$ and $\lim_{z\to+\infty} \psi(z) = C$. Let $\psi_n = \psi$ for all $n$. Then we can apply Theorem 27 and obtain $f = \psi^{**}$ and

$$g(z) = \begin{cases} 0 & \text{if } z \ge 0 \\ +\infty & \text{if } z < 0. \end{cases}$$

Note that $f(z) = 0$ if $z \ge 0$.

(iv) (scaled Lennard-Jones type potentials) Let $\psi$ be as in the previous example, and choose

$$\psi_n(z) = \frac{1}{\lambda_n}\psi(z).$$

Then we can apply Theorem 25 with

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

and $g$ as in Example (ii) above. In this case the limit energy $F$ is finite only on piecewise-constant functions with a finite number of positive jumps. On such functions $F(u) = C\#(S(u))$.

We now give an example which illustrates the effect of the operation of the subadditive envelope.

*Example 5.* If we take

$$\psi_n(z) = z^2 \wedge \Big( \frac{1}{\lambda_n} + (|z|\sqrt{\lambda_n} - 1)^2 \Big)$$

with $\lambda_n$ converging to 0, then we obtain $f(z) = z^2$ and

$$g(z) = \text{sub}^-(1 + z^2) = \min\Big\{ k + \frac{z^2}{k} \,:\, k = 1, 2, \dots \Big\}.$$

**A remark on second-neighbour interactions**

Consider functionals of the form

$$E_n(u) = \sum_i \lambda_n \psi_n^1 \Big( \frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n} \Big) + \sum_i 2\lambda_n \psi_n^2 \Big( \frac{u(x_n^{i+2}) - u(x_n^i)}{2\lambda_n} \Big). \quad (167)$$

If both sequences of functions $(\psi_n^i)_n$ satisfy conditions of Corollary 2 and some additional growth conditions from above, then it can be seen that the conclusions of Theorem 25 hold with

$$f(z) = \lim_n \Big( \psi_n^1(z) + 2\psi_n^2(z) \Big),$$

and

$$g(z) = \lim_n \lambda_n \Big( \psi_n^1 \Big( \frac{z}{\lambda_n} \Big) + 4\psi_n^2 \Big( \frac{z}{2\lambda_n} \Big) \Big).$$

This means that $E_n$ can be decomposed as the sum of three 'nearest-neighbour type' functionals, with underlying lattices $\lambda_n\mathbb{Z}$, $2\lambda_n\mathbb{Z}$ and $\lambda_n(2\mathbb{Z}+1)$, respectively, whose $\Gamma$-convergence can be studied separately. We now show that a similar conclusion does not hold if we remove the convexity/concavity hypothesis on $\psi_n^i$.

*Example 6.* Let $(\lambda_n)$ be a sequence of positive numbers converging to 0, and let $M > 2$ be fixed. Let $E_n$ be given by (167) with

$$\psi_n^k(z) = \begin{cases} z^2 & \text{if } |z| \le 1/\sqrt{k\lambda_n} \\ \frac{1}{k\lambda_n} g^k \Big( k\lambda_n z \Big) & \text{if } |z| > 1/\sqrt{k\lambda_n} \end{cases}$$

$(k = 1, 2)$, where

$$g^1(z) = \begin{cases} M & \text{if } |z| < 8 \\ 1 & \text{if } |z| \geq 8 \end{cases} \qquad g^2(z) = \begin{cases} 1 & \text{if } |z| \leq 1 \\ M & \text{if } |z| > 1. \end{cases}$$

Neither $g^i$ is subadditive and we have

$$\mathrm{sub}^- g^1(z) = \begin{cases} 2 & \text{if } |z| < 8 \\ 1 & \text{if } |z| \geq 8 \end{cases} \qquad \mathrm{sub}^- g^2(z) = \begin{cases} 1 & \text{if } |z| \leq 1 \\ 2 & \text{if } |z| > 1. \end{cases}$$

We can view $E_n$ as the sum of a first-neighbour interaction functional and two second-neighbour interaction functionals, to whom we can apply separately Theorem 25, obtaining the limit functionals

$$F^1(u) = \int_a^b |\dot{u}|^2 \, dt + \sum_{S(u)} \mathrm{sub}^- g^1([u])$$

for the first, and

$$F^2(u) = \int_a^b |\dot{u}|^2 \, dt + \sum_{S(u)} \mathrm{sub}^- g^2([u])$$

for each of the second ones. We will show that the $\Gamma$-limit of $E_n$ is strictly greater than $F^1(u) + 2F^2(u)$ at some $u \in SBV(a, b)$.

Let $u$ be given simply by $u = \chi_{(t_0, b)}$ with $t_0 \in (a, b)$. In this case $F^1(u) + 2F^2(u) = 4$. Suppose that there exist $u_n \in \mathcal{A}_n(a, b)$ converging to $u$ and such that $\limsup_n E_n(u_n) \leq 4$. In this case it can be easily seen that for $n$ large enough there must exist $i_n$ such that

$$u_n(x^{i_n}) - u_n(x^{i_n - 1}) > 4, \qquad u_n(x^{i_n + 1}) - u_n(x^{i_n}) < -4,$$

but

$$|u_n(x^{i_n - 1}) - u_n(x^{i_n - 2})| < 1, \qquad |u_n(x^{i_n + 2}) - u_n(x^{i_n + 1})| < 1.$$

This implies that

$$u_n(x^{i_n}) - u_n(x^{i_n - 2}) > 3, \qquad u_n(x^{i_n + 2}) - u_n(x^{i_n}) < -3,$$

so that $\limsup_n E_n(u_n) \geq 2M$, which gives a contradiction.

### 3.3 Long-range interactions

We conclude this chapter with a general statement whose proof can be obtained by carefully using the arguments of Section 1.4 and of the previous sections in this chapter. We use the notation of Chapter 1.

Let $K \in \mathbb{N}$ be fixed. For all $n \in \mathbb{N}$ and $j \in \{1, \dots, K\}$ let $\psi_n^j : \mathbb{R} \to (-\infty, +\infty]$ be given Borel functions bounded below. Define $E_n : L^1(0, L) \to [0, +\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \left( \frac{u(x_n^{i+j}) - u(x_n^i)}{j\lambda_n} \right) & x \in \mathcal{A}_n(0, L) \\ +\infty & \text{otherwise in } L^1(0, L). \end{cases} \tag{168}$$

We will describe the asymptotic behaviour of $E_n$ as $n \to +\infty$ when the energy densities are potentials of Lennard-Jones type. More precisely, we will make the following assumptions.

(H1) (*growth conditions*) There exists a convex function $\Psi : \mathbb{R} \to [0, +\infty]$ and $p > 1$ such that

$$\lim_{z \to -\infty} \frac{\Psi(z)}{|z|} = +\infty$$

and there exist constants $c_j^1$, $c_j^2 > 0$ such that

$$c_j^1(\Psi(z) - 1) \leq \psi_n^j(z) \leq c_j^2 \max\{\Psi(z), |z|\}$$

for all $z \in \mathbb{R}$.

*Remark 19.* Hypothesis (H1) is designed to cover the case of Lennard-Jones potentials (and potential of the same shape. Another case included in hypotheses (H1) is when all functions satisfy a uniform growth condition of order $p > 1$; *i.e.*,

$$(|z|^p - 1) \leq \psi_n^j(z) \leq c(|z|^p + 1)$$

for all $j$ and $n$.

Before stating our main result, we have to introduce the counterpart of the energy densities $f_n$ and $g_n$ in the previous section for the case $K > 1$. The idea is roughly speaking to consider clusters of $N$ subsequent points ($N$ large) and define an average discrete energy for each of those clusters, so that the energy $E_n$ may be approximately regarded as a 'nearest neighbour interaction energy' acting between such clusters, to which the above description applies.

We fix a sequence $(N_n)$ of natural numbers with the property

$$\lim_n N_n = +\infty, \qquad \lim_n \frac{N_n}{n} = 0. \tag{169}$$

We define

$$\psi_n(z) = \min\Big\{ \frac{1}{N_n} \sum_{j=1}^{K} \sum_{i=0}^{N_n - j} \psi_n^j\Big(\frac{u(i+j) - u(i)}{j}\Big) : \; u : \{0, \dots, N_n\} \to \mathbb{R}, \\ u(x) = zx \text{ if } x = 0, \dots, K, N_n - K, \dots, N_n \Big\}. \tag{170}$$

By using the energies $\psi_n$ we will regard a system of $N_n$ neighbouring points as a single interaction between the two extremal ones, up to a little error which is negligible as $N_n \to +\infty$. We can now state our convergence result, whose thesis is exactly the same as that of Theorem 27 upon replacing $\lambda_n$ by $\varepsilon_n := N_n\lambda_n$.

**Theorem 28.** *Let $\psi_n^j$ satisfy* (H1) *and let $(E_n)$ be given by* (168)*. Let $\psi_n$ be given by* (170) *and let $\varepsilon_n = N_n\lambda_n$. For all $n \in \mathbb{N}$ let $T_n \in \mathbb{R}$ be defined as in* (162)*, and let $f_n, g_n : \mathbb{R} \to [0, +\infty]$ be defined by*

$$f_n(z) = \begin{cases} \psi_n(z) & z \le T_n \\ +\infty & z > T_n \end{cases} \tag{171}$$

$$g_n(z) = \begin{cases} \varepsilon_n \psi_n\left(\dfrac{z}{\varepsilon_n}\right) & \text{if } z > \varepsilon_n T_n \\ +\infty & \text{otherwise.} \end{cases} \tag{172}$$

*Assume that there exist $f, g : \mathbb{R} \to [0, +\infty]$ such that*

$$\Gamma\text{-}\lim_n f_n^{**} = f \text{ on } \mathbb{R}, \tag{173}$$

$$\Gamma\text{-}\lim_n \operatorname{sub}^- g_n = g \text{ on } \mathbb{R} \setminus \{0\}. \tag{174}$$

*Note that this assumption is always satisfied, upon extracting a subsequence. Then, $(E_n)_n$ $\Gamma$-converges to $F$ with respect to the convergence in $L^1_{\mathrm{loc}}(0, L)$ and the convergence in measure, where*

$$F(u) = \begin{cases} \displaystyle\int_0^L \overline{f}(\dot u)\, dx + \sum_{S(u)} \overline{g}([u]) + \sigma Du_c^+(0, L) & \text{if } u \in BV_{\mathrm{loc}}(0, L)\ D_c u^- = 0 \\ & \text{and } [u] > 0 \text{ on } S(u) \\ +\infty & \text{otherwise in } L^1(0, L). \end{cases}$$

*where $\overline{f}$ and $\overline{g}$ are defined by (for notational convenience we set $g(0) = 0$)*

$$\overline{f}(z) := \inf\{f(z_1) + g^0(z_2) : z_1 + z_2 = z\},$$

$$\overline{g}(z) := \inf\{f^\infty(z_1) + g(z_2) : z_1 + z_2 = z\},$$

*and $\sigma := \overline{f}^\infty(1)$.*

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# Relaxation for bulk and interfacial energies

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## 1 Introduction

Several physical phenomena in phase transitions, fracture mechanics, image segmentation may be formulated by a variational model; this implies the study of the existence of equilibria, and hence of the existence of minimizers, for the related energy functional. This will be done by using the so called direct method of the Calculus of Variations based on lower semicontinuity and coerciveness of the energy functional. It is clear that more general lower semicontinuous results allow to consider more general variational models, and when the lower semicontinuity is not available, the characterization of the associated relaxed functional allows to understand the behaviour of minimizing sequences of the functional we started with.

In these lectures we will present some lower semicontinuity and relaxation results for functionals of the form

$$F(u) = \int_{\Omega} f(\nabla u)\, dx + \int_{S_u \cap \Omega} g(u^+, u^-, \nu)\, dH^{n-1}\,, \tag{1}$$

where $u$ is a “smooth” function outside a $(n-1)$-dimensional discontinuity set $S_u$ on which the traces $u^+$ and $u^-$ and the normal $\nu$ are well defined. Our main goal will be

(i) the search for (necessary and) sufficient conditions on $f$ and $g$ which will guarantee lower semicontinuity of the functional $F$ in some natural function space;
(ii) the study of the relaxed energy, and the identification of the relaxed functional in some integral form, when the lower semicontinuity of $F$ fails.

Minimum problems associated to (1) are named, after De Giorgi, “free discontinuity problems” since together with a standard volume part we have a surface energy term, concentrated on $(n-1)$-dimensional sets. An important feature of these problems lies in the fact that the support of the surface energy is not fixed a priori, and is usually an unknown of the problem.

Let us mention just two of the principal examples of variational problems that can be described in this setting.

Let us consider an elastic body which under the action of force fields or subjected to boundary displacements may fracture in some (a priori unknown) region. Here $\Omega \subseteq \mathbb{R}^n$ is a bounded open domain ($n = 3$ in the applications) and represents the reference configuration of the body, and $u : \Omega \to \mathbb{R}^n$ stands for the displacement field. The corresponding energy functional is of the form (1): $f(\nabla u)$ is the bulk energy density relative to the elastic deformation outside the fracture, whereas $g(u^+, u^-, \nu)$ is the surface energy density on the fracture $S_u$ (necessary to produce the crack).

In the framework of Griffith's materials, the energy necessary to the production of the crack is proportional to the crack surface, that is, $g(u^+, u^-, \nu) = \alpha$; in this case (1) becomes of the special form

$$F(u) = \int_\Omega f(\nabla u)\, dx + \alpha H^{n-1}(S_u \cap \Omega)\,.$$

Different models have been proposed to allow an interaction between the two sides of the crack opening; this can be done by considering, in the isotropic case, functionals like

$$F(u) = \int_\Omega f(\nabla u)\, dx + \int_{S_u \cap \Omega} g(|u^+ - u^-|)\, dH^{n-1}$$

with $g(t) \to 0$ as $t \to 0$ (Barenblatt's theory).

A second important example of functional of the form (1) is the leading part of the weak formulation in $SBV$ of the Mumford-Shah functional for the segmentation problem given by

$$G(u) = \int_{\Omega \setminus S_u} |\nabla u|^2\, dx + \alpha H^{n-1}(S_u \cap \Omega) + \beta \int_\Omega (u - g)^2\, dx\,.$$

As above $\Omega$ is a bounded open set in $\mathbb{R}^n$, $\alpha$ and $\beta$ are fixed positive parameters, and $g \in L^\infty(\Omega)$. When $n = 2$ and $g$ represents the 2-dimensional image given by a camera, then the minimization problem associated to $G$ corresponds to the search for a "smoothed" approximated image $u$ outside the set of contours $S_u$.

For a detailed discussion of the previous functional and other examples of variational problems that can be formulated and studied in this setting we refer to the book by [13], Section 4 and Section 6.

Let us briefly describe the functional framework we have to consider in order to define in a correct way the functional (1). The natural function space of weakly differentiable functions allowing the presence of discontinuity sets is the space $BV(\Omega; \mathbb{R}^k)$ of functions of bounded variation. For every $u \in BV(\Omega; \mathbb{R}^k)$ the quantities $S_u$, $u^+$, $u^-, \ldots$ involved in our above description (see Sections 4 and 7) are well defined. Moreover, the measure $Du$ can be decomposed as

$$Du = \nabla u L^n + (u^+ - u^-) \otimes \nu H^{n-1} \llcorner S_u + D_c u\,,$$

where $\nabla u$ denotes the density of the absolutely continuous part of $Du$ with respect to the Lebesgue measure, $(u^+ - u^-) \otimes \nu H^{n-1} \llcorner S_u$ is the Hausdorff part (jump part) and $D_c u$ is the so called Cantor part of $Du$, concentrated on a $L^n$ negligible set and such that $(D_c u)(B) = 0$ whenever $H^{n-1}(B) < +\infty$. We denote by $SBV(\Omega;\mathbb{R}^k)$ the class of special functions of bounded variation, i.e., $u$ belongs to $SBV(\Omega;\mathbb{R}^k)$ if $u \in BV(\Omega;\mathbb{R}^k)$ and $D_c u = 0$. This latter space is the right environment to define the functional (1). However, there are significant cases (e.g., if $g(a, b, \nu)$ behaves like a multiple of $|a - b|$, for $a$ close to $b$) where the domain of the relaxed functional is larger than $SBV$ (see, for the 1-dimensional case Theorem 13, for then n-dimensional case Theorem 17, Theorem 21 and Theorem 25).

Let us come now to the plan of these lectures. In the first part (corresponding to Sections 2-4) we recall the basic notation and give the main ingredients we will use afterwards, such as lower semicontinuity, the direct method of the Calculus of Variations and relaxation, as well as the notion and properties of functions of (special) bounded variation.

Then we state some suitable versions of general lower semicontinuity theorems and prove relaxation results for functionals of type (1) in dimension 1. These will be subsequently extended to higher dimension as well as for vector-valued functions.

Finally, let us point out that in the 1-dimensional case the relaxation results can be proved 'by hand' whereas in the higher dimensional case one has to use fine properties of $BV$ functions as the co-area formula. Since such formula is not available for vector-valued $BV$-functions, we have to proceed in a completely different and more technical manner in order to proof the main relaxation theorem of Section 7 (see also Section 8).

## 2 Lower semicontinuity - The direct method of the Calculus of Variations

The direct method of the Calculus of Variations can be summarized in the equation

$$\boxed{\text{Lower Semicontinuity + Coerciveness} \Longrightarrow \text{Existence of minimizers}}$$

meaning that, in order to obtain the existence of minimizers for a certain functional, it is sufficient to find a suitable topology in which a compact set can be found where a minimizing sequence lies (coerciveness), and at the same time the functional under consideration is lower semicontinuous.

In this section we recall briefly the notions of lower semicontinuity and coerciveness, and state the direct method of the Calculus of Variations (for more details and proofs we refer to the books [33],[27],[23]).

Let $X$ be a metric space, and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Let $F : X \to \overline{\mathbb{R}}$ be a function; we use the notation $\{F \leq t\} = \{x \in X : F(x) \leq t\}$. The level sets $\{F < t\}$, $\{F \geq t\}$, $\{F > t\}$ are defined in a similar way.

**Lower semicontinuity**

**Definition 1.** *A function $F : X \to \overline{\mathbb{R}}$ is said to be* (sequentially) lower semicontinuous (l.s.c. for short) at $x$*, if*

$$F(x) \leq \liminf_{h \to +\infty} F(x_h)$$

*for every sequence $(x_h)$ converging to $x$, or in other words*

$$F(x) = \min\{\liminf_{h \to +\infty} F(x_h) : x_h \to x\}\,.$$

We will say that $F$ is lower semicontinuous (on $X$) if it is lower semicontinuous at all $x \in X$.

*Remark 1.* (i) If $F$ and $G$ are l.s.c. at $x$, then so is $F + G$.

(ii) Let $\{F_i : i \in I\}$ be an arbitrary family of l.s.c. functions. Then $F(x) = \sup_{i \in I} F_i(x)$ is l.s.c.

(iii) If $F = \mathbf{1}_E$ is the characteristic function of the set $E$ (see (6)) then $F$ is l.s.c. if and only if $E$ is open.

**Coerciveness condition**

**Definition 2.** *We say that a subset $K$ of $X$ is* (sequentially) compact *if every sequence in $K$ has a subsequence which converges to a point of $K$, i.e.,*

$$\forall (x_h) \subset K \qquad \exists x \in K\,, \quad \exists (x_{h_k}) \ : x_{h_k} \to x\,.$$

**Definition 3.** *A function $F : X \to \overline{R}$ is* (sequentially) coercive *if $\overline{\{F \leq t\}}$ is compact in $X$ for every $t \in \mathbb{R}$, i.e., the closure of $\{F \leq t\}$ is compact in $X$ for every $t \in \mathbb{R}$.*

We are now in a position to describe Tonelli's direct method for proving existence results in the Calculus of Variations.

Let $F : X \to \overline{R}$ be a function. A *minimum point* (or *minimizer*) for $F$ in $X$ is a point $x \in X$ such that $F(x) \leq F(y)$ for every $y \in X$, i.e.,

$$F(x) = \inf_{y \in X} F(y)\,.$$

A *minimizing sequence* for $F$ in $X$ is a sequence $(x_h)$ in $X$ such that

$$\inf_{y\in X} F(y) = \lim_{h\to+\infty} F(x_h)\,.$$

**Direct method - (Weierstrass theorem)**

**Theorem 1.** *Let $F : X \to \overline{R}$ be coercive and lower semicontinuous. Then*

*(i) $F$ has a minimum point in $X$.*
*(ii) If $(x_h)$ is a minimizing sequence of $F$ in $X$, and $x$ is the limit of a subsequence of $(x_h)$, then $x$ is a minimum point of $F$ in $X$.*
*(iii) If $F$ is not identically $+\infty$, then every minimizing sequence for $F$ has a convergent subsequence.*

## 3 Relaxation

Let $F : X \to \overline{\mathbb{R}}$ be a coercive functional, but not lower semicontinuous. In general we can not say that $F$ has a minimum point on $X$.

Therefore we study the notion of relaxation, which allows to describe the minimizing sequences of functionals that are not lower semicontinuous in terms of minimum points of suitable lower semicontinuous functionals; in other words, we associate to $F$ a functional denoted by $\overline{F}$, which admits a minimum on $X$ and has the following properties:

(i) $\min_{x\in X} \overline{F}(x) = \inf_{x\in X} F(x)$;
(ii) the minimum points for $\overline{F}$ are the limits of minimizing sequences of $F$, and every minimizing sequence of $F$ has a subsequence which converges to a minimum point of $\overline{F}$.

**Definition 4.** *Let $F : X \to \overline{\mathbb{R}}$. The* lower semicontinuous envelope *(or* relaxed function*) $\overline{F}$ of $F$, denoted also by* $\mathrm{sc}^{-}F$*, is the greatest lower semicontinuous function less than or equal to $F$, i.e., for every $x \in X$*

$$\overline{F}(x) = \sup\{G(x) : G : X \to \overline{\mathbb{R}}\,,\ G \textit{ is l.s.c.}\,, G \le F\}\,. \tag{2}$$

By Remark 1 (ii) we have that (2) gives a lower semicontinuous function.

Let us point out that the function $\overline{F}$ can be characterized in terms of sequences as follows: for every $x \in X$

$$\begin{aligned}\overline{F}(x) &= \min\bigl\{\liminf_{h\to+\infty} F(x_h) : x_h \to x\bigr\}\\ &= \min\bigl\{\lim_{h\to+\infty} F(x_h) : x_h \to x\,, \text{and } \exists \lim_{h\to+\infty} F(x_h)\bigr\}\,.\end{aligned} \tag{3}$$

We consider now the connection between the minimum problem $\min_{x\in X} F(x)$ and the relaxed problem $\min_{x\in X} \overline{F}(x)$. In particular, the following theorem describes the behaviour of the minimizing sequences of $F$ in terms of the minimizers of $\overline{F}$.

**Theorem 2.** *Let $F : X \to \overline{\mathbb{R}}$ be coercive. Then the following properties hold:*

*(i)* $\overline{F}$ *is coercive and lower semicontinuous;*
*(ii)* $\overline{F}$ *has a minimum point in* $X$ ;
*(iii)* $\min_{x \in X} \overline{F}(x) = \inf_{x \in X} F(x)$ ;
*(iv) the minimum points for* $\overline{F}$ *are exactly all the limits of minimizing sequences of* $F$.

*Remark 2.* For functionals of the form

$$F(u) = \int_{\Omega} f(x, u, \nabla u)\, dx \tag{4}$$

defined on some Sobolev space $W^{1,p}(\Omega)$ ($p \geq 1$) the direct method introduced in Section 2 is useless whenever the integrand function $f$ has linear growth of the type

$$|z| \leq f(x, u, z) \leq c(1 + |z|)\,. \tag{5}$$

Indeed, even if $f$ satisfies all assumptions of the lower semicontinuity theorems, the coerciveness condition fails because the sets

$$\Big\{ u \in W^{1,p}(\Omega) : \int_{\Omega} |\nabla u|\, dx \leq t \Big\}$$

are not sequentially relatively compact on $W^{1,p}(\Omega)$ endowed with its weak topology or with the strong $L^p$-topology.

For this reason it is convenient to consider a space larger than $W^{1,p}(\Omega)$ in order to define functionals with linear growth in the gradient. This space will be the space $BV(\Omega)$ of functions with bounded variation.

Working with the space $BV(\Omega)$, from (5) it follows that every minimizing sequence $(u_h)$ of $F$ will be bounded in $BV(\Omega)$ and converges therefore, up to a subsequence, in $L^1$ to a $BV$ function $u$ (using a compactness theorem on $BV(\Omega)$). It is then natural to extend the functional $F$ by relaxation with respect to the $L^1$-topology to $BV(\Omega)$ (for the relaxed functional $\overline{F}$ we have then lower semicontinuity and coerciveness!). The main problem is then to find a representation formula for $\overline{F}$. This problem has been studied by many mathematicians.

The aim of these lectures (see Sections 5-8) is to show how to tackle with the problem of representation of $\overline{F}$, when $F$ has not only the volume part (i.e., is not only of the form (4)), but has also a surface energy term in its natural setting of $BV$ functions.

## 4 Preliminaries. Functions of bounded variation and special functions of bounded variation

### 4.1 Notation

Let us collect here briefly some notation used frequently in the following sections. If some notation is used only “locally” we will it introduce near to its use.

Let $n\,, k \in \mathbb{N}$ be fixed. We shall denote by $I$ an open interval of $\mathbb{R}$, and by $\Omega$ a (bounded) open subset of $\mathbb{R}^n$. Let us denote by $A(\Omega)$ (resp. $B(\Omega)$) the family of the open (resp. Borel) subsets of $\Omega$. We shall use standard notations for the Sobolev and Lebesgue spaces $W^{1,p}(\Omega;\mathbb{R}^k)$ and $L^p(\Omega;\mathbb{R}^k)$. When $k=1$, we shall drop the target space $\mathbb{R}^k$ in the notation, and write just $W^{1,p}(\Omega)$, $L^p(\Omega)$, and so one. In particular we will denote by $\|u\|_p$ the $L^p$-norm of a function $u$ in $L^p(\Omega)$.

The Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ will be denoted by $L^n$ and $H^{n-1}$, respectively. We shall use also the notation $|E|$ for $L^n(E)$, the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$, and $\#$ for $H^0$, the counting measure.

Let us point out that $|\cdot|$ will be used also to denote the usual euclidean norm on $\mathbb{R}^n$ or the total variation of a measure (see below). The correct interpretation will be clear from the context or will be explained otherwise. We will denote by $\langle\cdot,\cdot\rangle$ the usual scalar product on $\mathbb{R}^n$.

Let $X$ be a set, and $E \subset X$; we define the *characteristic function* of $E$ as

$$\mathbf{1}_E(z) = \begin{cases} 1 \text{ if } z \in E \\ 0 \text{ if } z \in X \setminus E, \end{cases} \tag{6}$$

and the *indicator function* of $E$ as

$$\chi_E(z) = \begin{cases} 0 & \text{if } z \in E \\ +\infty & \text{if } z \in X \setminus E. \end{cases} \tag{7}$$

If $N \geq 1$ is an integer, then $\phi : \mathbb{R}^N \to [\,0,+\infty\,]$ is *positively homogeneous of degree one* if $\phi(\lambda\xi) = \lambda\phi(\xi)$ for every $\lambda > 0$ and for every $\xi \in \mathbb{R}^N$.

Let $f : \mathbb{R}^N \to [\,0,+\infty\,]$ be a convex function. We define the *recession function* $f^\infty$ of $f$ as

$$f^\infty(\xi) = \lim_{t\to+\infty} \frac{f(t\xi)}{t} \qquad \text{for every } \xi \in \mathbb{R}^N. \tag{8}$$

It is well known (see [43] that this limit exists, and that $f^\infty$ is a Borel function, which is convex and positively homogeneous of degree one. If $f(0)=0$, then $f(\xi) \leq f^\infty(\xi)$ for every $\xi \in \mathbb{R}^N$. Indeed, by the convexity assumption of $f$ we get for $\xi \in \mathbb{R}^N$ and $t > 0$

$$f(\xi) = f\big(t\frac{\xi}{t} + (1-t)0) \leq tf(\frac{\xi}{t})$$

and by passing to the limit as $t \to +\infty$ we get $f(\xi) \le f^\infty(\xi)$.

If $\phi : \mathbb{R}^N \to [\,0, +\infty\,]$ is a Borel function, then we denote by $\phi^{**}$ the *convex and lower semicontinuous envelope of* $\phi$, i.e., the greatest convex and lower semicontinuous function less than or equal to $\phi$ (see, for instance, [36]). If $\phi$ is finite and continuous then the following equality holds

$$\phi^{**}(\xi) = \inf\Big\{ \sum_{i=1}^{N+1} \lambda_i \phi(\xi_i) : \lambda_i \ge 0 \,, \sum_{i=1}^{N+1} \lambda_i = 1 \,, \sum_{i=1}^{N+1} \xi_i = \xi \Big\}$$

for every $\xi \in \mathbb{R}^N$.

Let $\theta : \mathbb{R} \to [\,0, +\infty[$ be a subadditive function, i.e., $\theta(x+y) \le \theta(x) + \theta(y)$ for every $x$, $y \in \mathbb{R}$. Assume $\theta(0) = 0$. We can define the function $\theta^0 : \mathbb{R} \to [\,0, +\infty\,]$ (see, for instance, [15]) by

$$\theta^0(z) = \lim_{t \to 0^+} \frac{\theta(tz)}{t} = \sup_{t>0} \frac{\theta(tz)}{t} \,. \tag{9}$$

The function $\theta^0$ is convex, subadditive, and positively homogeneous of degree one (it follows that $\theta^0(z) = z\theta^0(1)$, where $\theta^0(1)$ is the slope (from the right-hand side) of $\theta$ in 0). Moreover, $\theta^0 \ge \theta$ (it follows easily by $\theta^0(z) = \sup_{t>0} \frac{\theta(tz)}{t} \ge \theta(1z)$).

The letters $c$, $c'$, $\ldots$, $c_1$, $\ldots$ will denote always a strictly positive constant, whose value may vary from line to line, and is always independent of the parameters of the problems considered each time.

### 4.2 Measure spaces

Let us recall first the main definitions of measure theory and collect the main results we will use later.

**Definition 5.** *A function* $\mu : B(\Omega) \to \mathbb{R}^N$ *is a* (vector) measure *on* $\Omega$ *if* $\mu(\emptyset) = 0$ *and if it is countably additive; i.e.,*

$$B = \bigcup_{i \in \mathbb{N}} B_i \qquad B_i \cap B_j = \emptyset \quad \text{if } i \ne j \quad \Longrightarrow \qquad \mu(B) = \sum_{i \in \mathbb{N}} \mu(B_i) \,.$$

*The set of such measures will be denoted by* $M(\Omega; \mathbb{R}^N)$*. If no confusion may arise, we denote by* $\mu_i$ $(i = 1, \ldots, N)$ *the components of* $\mu$.

*We say that a measure is a* scalar measure *if* $N = 1$*, and that it is a* positive measure *if it takes values in* $[\,0, +\infty\,]$*. The sets of scalar and positive measures will be denoted by* $M(\Omega)$ *and* $M_+(\Omega)$*, respectively.*

*A function* $\mu : B_c(\Omega) \to \mathbb{R}^N$ *(*$B_c(\Omega)$*= family of Borel subsets of* $\Omega$ *with compact closure) is a* Radon measure *on* $\Omega$ *if* $\mu|_{B(\Omega')}$ *is a measure on* $\Omega'$ *for all* $\Omega' \subset\subset \Omega$*. As above, we will speak of scalar and of positive Radon measures.*

*If $\mu \in M(\Omega)$, $L^p(\Omega, \mu; \mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued $p$-summable functions with respect to $\mu$ on $\Omega$ (when $\mu = L^n$ we omit it; see above).*

*If $\mu \in M(\Omega; \mathbb{R}^N)$ and $B \in B(\Omega)$, we define $(\mu \llcorner B)(A) = \mu(B \cap A)$ for $A \in B(\Omega)$. We have that $\mu \llcorner B \in M(\Omega; \mathbb{R}^N)$.*

*If $\mu \in M(\Omega; \mathbb{R}^N)$ for all $B \in B(\Omega)$ we define the* total variation *of $\mu$ on $B$ by*

$$|\mu|(B) = \sup\Big\{\sum_{i\in\mathbb{N}} |\mu(B_i)| : B = \bigcup_{i\in\mathbb{N}} B_i\,, B_i \textit{ pairwise disjoint}\Big\}\,.$$

*The set function $|\mu|$ is a positive finite measure on $\Omega$ (see [13], Theorem 1.6). The support of $\mu \in M(\Omega; \mathbb{R}^N)$ is defined as*

$$\mathrm{spt}\mu = \{x \in \Omega : |\mu|(B_\rho(x)) > 0 \ \ \forall B_\rho(x) \subset \Omega\}\,.$$

**Definition 6.** *Let $\mu \in M_+(\Omega)$ and $\lambda \in M(\Omega; \mathbb{R}^N)$. We say that $\lambda$ is* absolutely continuous *with respect to $\mu$ (and we write $\lambda \ll \mu$) if $|\lambda|(B) = 0$ for every $B \in B(\Omega)$ with $\mu(B) = 0$.*

*We say that $\lambda$ is* singular *with respect to $\mu$ if there exists a set $E \in B(\Omega)$ such that $\mu(E) = 0$ and $|\lambda|(B) = 0$ for all $B \in B(\Omega)$ such that $B \cap E = \emptyset$ (we say that $\lambda$ is* concentrated *on $E$).*

*Remark 3.* If $f \in L^1(\Omega, \mu; \mathbb{R}^N)$ and $\mu \in M(\Omega)$ then we define the measure $f\mu \in M(\Omega; \mathbb{R}^N)$ by

$$f\mu(B) = \int_B f\, d\mu\,.$$

We have that $f\mu \ll |\mu|$. Moreover $|f\mu| = |f|\,|\mu|$.

**Theorem 3 (Radon-Nikodým).** *If $\lambda \in M(\Omega; \mathbb{R}^N)$ and $\mu \in M_+(\Omega)$, then there exists a function $f \in L^1(\Omega, \mu; \mathbb{R}^N)$ and a measure $\lambda_s$, singular with respect to $\mu$, such that*

$$\lambda = f\,\mu + \lambda_s\,, \qquad \textit{with } \ f(x) = \lim_{\rho\to 0^+} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))} \ \ \textit{for } \mu\textit{-a.e. } x \in \Omega.$$

*This will be called the* Radon-Nikodým decomposition *of $\lambda$ with respect to $\mu$, and the function $f$ (given by the Besicovitch derivation theorem (see [13])) the* Radon-Nikodým derivative *of $\lambda$ with respect to $\mu$, and denoted by $\dfrac{d\lambda}{d\mu}$.*

*Remark 4.* From the previous theorem we get the following results:

(a) If $\lambda \ll \mu$, then $\lambda = f\,\mu$ for some $f \in L^1(\Omega, \mu; \mathbb{R}^N)$;
(b) Since $\lambda \ll |\lambda|$ there exists $\nu \in L^1(\Omega, |\lambda|; \mathbb{R}^N)$ such that

$$\lambda = \nu|\lambda|\,.$$

Since $|\lambda| = |\nu|\lambda|\,| = |\nu|\,|\lambda|$ we get that $|\nu| = 1$ $|\lambda|$-a.e. on $\Omega$.

Let us consider the space $C_0(\Omega;\mathbb{R}^N)$ of all continuous functions from $\Omega$ into $\mathbb{R}^N$ which vanish on the boundary; that is, $\phi \in C_0(\Omega;\mathbb{R}^N)$ if $\forall\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $|\phi| < \varepsilon$ on $\Omega \setminus K_\varepsilon$. By Riesz's theorem we get $M(\Omega;\mathbb{R}^N) = C_0(\Omega;\mathbb{R}^N)'$ with the duality $\langle \mu, \phi\rangle = \int_\Omega \phi\, d\mu$ so that

$$|\mu|(\Omega) = \sup\Big\{ \int_\Omega \phi\, d\mu : \phi \in C_0(\Omega;\mathbb{R}^N)\,,\, \|\phi\|_\infty \le 1 \Big\}\,.$$

The usual weak* topology on $M(\Omega;\mathbb{R}^N)$ is defined as the weakest topology on $M(\Omega;\mathbb{R}^N)$ for which the maps $\mu \mapsto \displaystyle\int_\Omega \phi\, d\mu$ are continuous for every $\phi \in C_0(\Omega;\mathbb{R}^N)$.

### 4.3 Functions of bounded variation

We begin this section with the most common definition of $BV(\Omega)$, based on the existence of a measure distributional derivative.

**Definition 7.** *Let $u \in L^1(\Omega)$. We say that $u$ is a* function of bounded variation *on $\Omega$ if its distributional derivative is a measure, i.e., there exists $\mu \in M(\Omega;\mathbb{R}^n)$ such that*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi\, d\mu_i \qquad \forall \phi \in C^1_c(\Omega)\,.$$

We denote $\mu = (\mu_1, \ldots, \mu_n) \doteq Du$ and its components by $D_i u$. The vector space of all functions of bounded variation in $\Omega$ is denoted by $BV(\Omega)$. Recall that $C^1_c(\Omega)$ denotes the space of $C^1$ functions with compact support in $\Omega$.

**Note.** The Sobolev space $W^{1,1}(\Omega) \subset BV(\Omega)$; indeed, for any $u \in W^{1,1}(\Omega)$ the distributional derivative is given by $\nabla u L^n$, and $|Du|(\Omega) = \int_\Omega |\nabla u|\, dx$. The inclusion is strict: there exist functions $u \in BV(\Omega)$ such that $Du$ is singular with respect to $L^n$ (for instance, the Heaviside function $\mathbf{1}_{(0,+\infty)}$, whose distributional derivative is the Dirac measure $\delta_0$).

One of the main advantages of the $BV$ space is that it includes, unlike Sobolev spaces, characteristic functions of sufficiently regular sets, and more generally, piecewise smooth functions.

The following approximation result is useful since it often enables to restrict the proofs to smooth functions.

**Theorem 4.** *[Approximation by smooth functions] Let $u \in BV(\Omega)$. Then there exists a sequence $(u_h) \subset C^\infty(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$ and $|Du_h|(\Omega) \to |Du|(\Omega)$.*

**Theorem 5.** *The following statements are equivalent:*

*(i)* $u \in BV(\Omega)\,;$

*(ii)* $u \in L^1(\Omega)$ *and the* total variation *of* $u$ *on* $\Omega$

$$V(u,\Omega) = \sup\Big\{\int_\Omega u \,\mathrm{div} g \, dx : g \in C^1_c(\Omega;\mathbb{R}^n)\,, \|g\|_\infty \le 1\Big\}$$

*is finite;*

*(iii) there exists a sequence of functions* $(u_h)$ *in* $C^\infty(\Omega)$ *such that* $u_h \to u$ *in* $L^1(\Omega)$ *and* $\limsup_{h\to+\infty} \int_\Omega |\nabla u_h| \, dx < +\infty$.

**Note.** It turns out that $V(u,\Omega) = |Du|(\Omega)$; motivated by this fact, also $|Du|(\Omega)$ will be sometimes called the variation of $u$ in $\Omega$.

**Note.** $BV(\Omega)$ endowed with the norm $\|u\|_{BV} \doteq \|u\|_1 + |Du|(\Omega)$ is a Banach space.

**Definition 8.** *Let* $u\,, u_h \in BV(\Omega)$. *We say that* $(u_h)$ weakly* converges *in* $BV(\Omega)$ *to* $u$ *if* $(u_h)$ *converges to* $u$ *in* $L^1(\Omega)$ *and* $(Du_h)$ *weakly* converges to* $Du$ *in* $\Omega$, *i.e.,*

$$\lim_{h\to+\infty} \int_\Omega \phi \, Du_h = \int_\Omega \phi \, Du \qquad \forall \phi \in C_0(\Omega)\,.$$

A simple criterion for weak* convergence is stated in the following proposition.

**Proposition 1.** *Let* $(u_h) \subset BV(\Omega)$. *Then* $(u_h)$ *weakly* converges to* $u$ *in* $BV(\Omega)$ *if and only if* $(u_h)$ *converges to* $u$ *in* $L^1(\Omega)$ *and* $(u_h)$ *is bounded in* $BV(\Omega)$.

*Remark 5.* If $u_h \to u$ in $L^1(\Omega)$, then $|Du|(\Omega) \le \liminf_{h\to+\infty} |Du_h|(\Omega)$.

As stated in Section 3 (Remark 2) the following compactness theorem for $BV$ functions is very useful in connection with variational problems with linear growth in the gradient. Since the Sobolev space $W^{1,1}$ has no a similar compactness property this provides also a justification for the introduction of $BV$ spaces in the Calculus of Variations (besides the necessity to handle in certain problems with possibly discontinuous functions).

**Theorem 6 (Compactness in $BV$).** *If* $(u_h) \subset BV(\Omega)$ *and* $\sup_{h\in\mathbb{N}} \|u_h\|_{BV} < +\infty$, *then there exists a subsequence of* $(u_h)$ *converging in* $L^1_{\mathrm{loc}}(\Omega)$ *to some* $u \in BV(\Omega)$.

*If* $\Omega$ *is an open set with compact Lipschitz boundary we can say that the subsequence weakly* converges to* $u$.

### 4.4 Sets of finite perimeter

**Definition 9.** *We say that a set $E \subset \mathbb{R}^n$ is a* set of finite perimeter *in $\Omega$ if $\mathbf{1}_E \in BV(\Omega)$; that is, if*

$$\sup\Big\{\int_E \operatorname{div} g\, dx : g \in C_c^1(\Omega;\mathbb{R}^n)\,,\, \|g\|_\infty \le 1\Big\} < +\infty\,.$$

*This quantity $|D\mathbf{1}_E|(\Omega)$ is called* the perimeter *of $E$ in $\Omega$.*

*Remark 6.* If $E \subset \mathbb{R}^n$ and $\partial E$ is piecewise $C^1$, then $|D\mathbf{1}_E|(\Omega) = H^{n-1}(\partial E \cap \Omega)$. If $E$ is of finite perimeter in $\Omega$ and $C \subset \Omega$ satisfies $H^{n-1}(C) = 0$, then $|D\chi_E|(C) = 0$.

Let $E$ be a set of finite perimeter in $\Omega$. The De Giorgi's *reduced boundary* of $E$, denoted by $\partial^* E$, is defined by

$$\partial^* E = \Big\{x \in \operatorname{spt}|D\mathbf{1}_E| : \exists \lim_{\rho \to 0^+} \frac{D\mathbf{1}_E(B_\rho(x))}{|D\mathbf{1}_E(B_\rho(x))|} \doteq \nu(x) \in S^{n-1}\Big\}\,.$$

The function $\nu : \partial^* E \to S^{n-1}$ is called the *interior normal* to $E$ ($S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$).

A set $S \subset \mathbb{R}^n$ is *rectifiable* if there exists a countable family $(\Gamma_i)$ of graphs of Lipschitz functions of $(n-1)$ variables such that $H^{n-1}(S \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$.

**Theorem 7 (De Giorgi's Rectifiability Theorem).** *Let $E$ be a set of finite perimeter in $\Omega$. Then*

*(i)* $\partial^* E$ *is rectifiable;*
*(ii)* $|D\mathbf{1}_E|(B) = H^{n-1}(\partial^* E \cap B)$. *In particular,* $H^{n-1}(\partial^* E) < +\infty$;
*(iii)* $D\mathbf{1}_E = \nu H^{n-1} \llcorner \partial^* E$.

**Theorem 8 (Co-area formula for $BV$ functions).** *If $u \in BV(\Omega)$, then the sets $\{u > t\}$ are of finite perimeter for a.e. $t \in \mathbb{R}$ and*

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} H^{n-1}(\partial^*\{u > t\} \cap \Omega)\, dt\,. \tag{10}$$

Note that this formula holds for all open sets $A \subset \Omega$ in place of $\Omega$, and since both sides define finite measures, we have

$$|Du|(B) = \int_{-\infty}^{+\infty} H^{n-1}(\partial^*\{u > t\} \cap B)\, dt$$

for all Borel sets $B \subset \Omega$, by regularity.

### 4.5 Structure of $BV$ functions. Approximate discontinuity points and approximate jump points

**Definition 10.** *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$. We say that $u$ has* approximate limit *at $x$ if there exists $z \in \mathbb{R}$ such that*

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y) - z| \, dy = 0 \, .$$

*The set $S_u$ where this property fails is called* approximate discontinuity set *of $u$. $z$ is uniquely determined for any point $x \in \Omega \setminus S_u$ and is called the* approximate limit *of $u$ at $x$ and denoted by $\widetilde{u}(x)$.*

If $E$ is a set of finite perimeter, we get

$$\partial^* E \cap \Omega = S_{\mathbf{1}_E} \cap \Omega \, .$$

If $\nu \in S^{n-1}$, we split any ball $B_\rho(x)$ into the two halves

$$\begin{aligned} B^+_\rho(x,\nu) &= \{y \in B_\rho(x) : \langle y - x, \nu \rangle > 0\} \\ B^-_\rho(x,\nu) &= \{y \in B_\rho(x) : \langle y - x, \nu \rangle < 0\} \, . \end{aligned}$$

**Definition 11.** *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$. We say that $x$ is an* approximate jump point *at $u$ if there exist $a, b \in \mathbb{R}$, and $\nu \in S^{n-1}$ such that $a \neq b$ and*

$$\lim_{\rho \to 0^+} \frac{1}{|B^+_\rho(x,\nu)|} \int_{B^+_\rho(x,\nu)} |u(y) - a| \, dy = 0 \, ,$$

$$\lim_{\rho \to 0^+} \frac{1}{|B^-_\rho(x,\nu)|} \int_{B^-_\rho(x,\nu)} |u(y) - b| \, dy = 0 \, .$$

*The set of approximate jump points (or* jump set*) of $u$ is denoted by $J_u$.*

The triplet $(a, b, \nu)$ which turns out to be uniquely determined up to a permutation of $a$ and $b$ and a change of sign of $\nu$, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. On $\Omega \setminus S_u$ we set $u^+ = u^- = \widetilde{u}$.

Let $u \in BV(\Omega)$. By the Radon-Nikodým theorem (Theorem 3) we set $Du = D_a u + D_s u$, where $D_a u$ is the *absolutely continuous part* of $Du$ with respect to the Lebesgue measure and $D_s u$ is the *singular part* of $Du$ with respect to $L^n$. If $D_j u = Du \llcorner S_u$ is the *jump part* of $Du$ and $D_c u = D_s u \llcorner (\Omega \setminus S_u)$ is the *Cantor part* of $u$ we can then write

$$Du = D_a u + D_j u + D_c u \, .$$

We have that $|D_c u|(B) = 0$ for all Borel sets $B$ such that $H^{n-1}(B) < +\infty$.

**Theorem 9 (Structure of $BV$-functions).** *Let $u \in BV(\Omega)$. Then*

*(i) the set $S_u$ is rectifiable and $H^{n-1}(S_u \setminus J_u) = 0$.*

*(ii) Moreover,*

$$D_a u = \frac{d\,D_a u}{d\,L^n} \doteq \nabla u\, L^n$$

$$D_j u = (u^+ - u^-)\nu_u H^{n-1} \llcorner J_u$$

*and $\nu_u(x)$ gives the approximate normal direction to $J_u$ for $H^{n-1}$ a.e. $x \in J_u$ ($\nabla u$ denotes the density of $D_a u$ with respect to $L^n$). Finally, we have*

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \langle \nabla u(x), y - x\rangle|}{|x - y|}\, dy = 0$$

*for a.e. $x \in \Omega$ (*approximate differentiability *of $u$ a.e. on $\Omega$).*

**Note:** Since for every $u \in BV(\Omega)$ we have $H^{n-1}(S_u \setminus J_u) = 0$, we will "identify" $S_u$ and $J_u$.

### 4.6 Special functions of bounded variation

The space $SBV(\Omega)$ can be defined as the space of the functions $u \in BV(\Omega)$ such that $D_c u = 0$. We have then

$$Du = \nabla\, L^n + (u^+ - u^-)\nu_u H^{n-1} \llcorner S_u\,.$$

It turns out that $SBV(\Omega)$ contains the bounded "piecewise" Sobolev functions. More precisely, if $\Omega$ is bounded, $K$ is a closed subset of $\mathbb{R}^n$ with $H^{n-1}(K \cap \Omega) < +\infty$ and $u \in L^\infty(\Omega)$ with $u \in W^{1,1}(\Omega \setminus K)$ then $u \in SBV(\Omega)$ and $H^{n-1}(S_u \setminus K) = 0$.

For a thorough treatment of $BV$ functions we refer to [13], [19].

### 4.7 *BV* (*SBV*) functions in one dimension

If $u \in BV(a, b)$ then it can be easily seen that there exists a constant $c$ such that

$$u(x) = c + Du(a, x)$$

by differentiating both sides of this equality in the sense of distributions. Moreover, the right-hand side and the left-hand side limits

$$u(t+) = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} u(s)\, ds \qquad\qquad u(t+) \doteq u^+$$

$$u(t-) = \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} u(s)\, ds \qquad\qquad u(t-) \doteq u^-$$

exist at all $t \in [\,a, b[$ and $t \in\, ]a, b]$, respectively. Finally, $u(t+) = u(t-)$ if $|Du|(\{t\}) = 0$.

If $M(I)$ is the space of all Radon measures on $I$ with bounded total variation and given $t_0 \in [\,a, b[$ and $u_0 \in \mathbb{R}$ (see above) there is a $1{-}1$ correspondence between $M(I)$ and the subspace $\{u \in BV(I) : u(t_0+) = u_0\}$ given by $\mu \mapsto u_\mu$, where

$$u_\mu(t) = \begin{cases} u_0 + \mu(\,]t_0, t\,]) \text{ if } t \geq t_0 \\ u_0 - \mu(\,]t, t_0\,]) \text{ otherwise}. \end{cases}$$

In treating functions in $BV(I)$ (for instance, in Section 5) we define sometimes functions in $BV(I)$ by simply describing their measure derivative and the value $u(t_0+)$ at some point $t_0 \in [\,a, b[$ (or equivalently $u(t_0-)$) at some point $t_0 \in \,]a, b\,]$.

The space $SBV(I)$ reduces to the set of functions $u$ in $BV(I)$ such that their measure first derivative, denoted in this case by $\dot{u}$, is of the form

$$\dot{u} = \dot{u}_a\, dt + \dot{u}_s = \dot{u}_a\, dt + \sum_{k=1}^{\infty} a_k \delta_{t_k}\,,$$

where $t_k \in I$, $a_k \in \mathbb{R}$, $\sum_{k=1}^{\infty} |a_k| < +\infty$, and $\delta_t$ is the Dirac measure at $t$. We get $a_k = u(t_k+) - u(t_k-)$.

Moreover, we will often express integration on $S_u$ as a summation: for example

$$\int_{S_u \cap \Omega} |u^+(t) - u^-(t)|\, dH^0(t) = \int_{S_u \cap \Omega} |u(t+) - u(t-)|\, d\#(t) =$$

$$= \sum_{t \in S_u \cap \Omega} |u(t+) - u(t-)|\,.$$

# 5 Lower semicontinuity and relaxation in one dimension

## 5.1 A characterization of l.s.c. for functionals defined on $SBV(I)$. Relaxation on $BV(I)$

Lower semicontinuity conditions for general functionals defined on $SBV$ take a complex form, due to the possible interaction of the Lebesgue part and the jump part. In this section we state a simplified version of a more general lower semicontinuity theorem in dimension one (see [19], Thm. 2.6).

**Theorem 10 (Characterization of l.s.c. for functionals defined on $SBV(I)$).** *Let $f : \mathbb{R} \to [\,0, +\infty[$ be convex and*

$$\lim_{|z| \to +\infty} \frac{f(z)}{|z|} = +\infty \qquad \textit{(i.e., } f \textit{ is superlinear at } \infty\textit{).} \tag{11}$$

*Let $\theta : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ satisfy* $\inf \theta > 0$. *Let $I \subset \mathbb{R}$ be a bounded open interval. Then the functional $F : BV(\Omega) \to [0, +\infty]$*

$$F(u) = \begin{cases} \displaystyle\int_I f(\dot{u}_a)\, dt + \sum_{t \in S_u} \theta(u(t+), u(t-)) & \text{if } u \in SBV(I), \\ +\infty & \text{if } u \in BV(I) \setminus SBV(I) \end{cases} \tag{12}$$

*is lower semicontinuous with respect to the $BV - w^*$ convergence in $BV(I)$ if and only if $\theta$ is lower semicontinuous and $\theta$ is subadditive, i.e.,*

$$\theta(x, y) \leq \theta(x, z) + \theta(z, y) \qquad \forall x, y, z \in \mathbb{R}\,. \tag{13}$$

*Remark 7.* Condition (11) can be substituted by the following one: there exists a convex function $\phi : \mathbb{R} \to [0, +\infty[$ with $\displaystyle\lim_{|z| \to +\infty} \frac{\phi(z)}{|z|} = +\infty$ and $f(z) \geq \phi(z)$ for every $z \in \mathbb{R}$.

*Remark 8.* By the superlinear assumption (11) and the convexity of $f$ we get easily that

$$f(z) \geq |z| - c \tag{14}$$

for some suitable constant $c \geq 0$. If $\theta(x, y) \geq \alpha|x - y|$ for $\alpha > 0$, $x, y \in \mathbb{R}$, then the above lower semicontinuity result in $BV - w^*$ is equivalent to lower semicontinuity of $F$ with respect to the $L^1$-convergence in $BV(I)$. Indeed, under the above assumptions we get that

$$F(u) \geq |\dot{u}|(I) - c'\,,$$

for some suitable constant $c'$ (by Theorem 6, this estimate implies that $L^1$ convergent sequences converge, up to a subsequence, weakly* in $BV(I)$).

*Remark 9.* Let us consider the special case of $\theta$ when

$$\theta(x, y) = \varphi(x - y)$$

with $\varphi : \mathbb{R} \to [0, +\infty[$. Then $\theta$ is subadditive, i.e., satisfies (13) if and only if

$$\varphi(x + y) \leq \varphi(x) + \varphi(y) \qquad \forall x, y \in \mathbb{R}\,. \tag{15}$$

If $\varphi$ satisfies (15) we will again say that it is subadditive.

**Examples:**

(i) Let $\varphi : [0, +\infty[ \to [0, +\infty[$ be concave. Then it is subadditive.
(ii) It can be easily shown that the functions $|\sin z|$, $\arctan |z|$, $\min\{|z|, 1\}$ define subadditive functions.
(iii) Let $\varphi : [0, +\infty[ \to [0, +\infty[$ be concave and non-decreasing. Then $\phi(z) = \varphi(|z|)$ is subadditive on $\mathbb{R}$.

Guided by the semicontinuity result above we can now state a relaxation result on $SBV(I)$ (see [21], Prop. 5.4). We shall focus our attention on the behaviour of the jump-part energy.

**Theorem 11 (Relaxation in $SBV(I)$ - no interaction of bulk and jump-part energy densities).** *Let $f : \mathbb{R} \to [\,0, +\infty[$ be a convex function, $f(0) = 0$ and*

$$|z|^2 - c \leq f(z) \leq c'(1 + |z|^2) \qquad \forall z \in \mathbb{R}\,. \tag{16}$$

*Let $\theta : \mathbb{R} \times \mathbb{R} \to [\,0, +\infty[$ be a Borel function satisfying*

$$\inf \theta > 0\,, \qquad \theta(x, y) \geq |x - y|\,. \tag{17}$$

*Let $F : BV(I) \to [\,0, +\infty\,]$ be the functional defined by*

$$F(u) = \begin{cases} \displaystyle\int_I f(\dot{u}_a)\,dt + \sum_{t \in S_u} \theta(u(t+), u(t-)) & \text{if } u \in SBV(I),\ \#(S_u) < +\infty, \\ +\infty & \text{otherwise on } BV(I). \end{cases} \tag{18}$$

*The relaxed functional with respect to the $L^1(I)$-topology, denoted by $\overline{F}$, can then be written as*

$$\overline{F}(u) = \begin{cases} \displaystyle\int_I f(\dot{u}_a)\,dt + \sum_{t \in S_u} \overline{\text{sub}}\,\theta(u(t+), u(t-)) & \text{if } u \in SBV(I), \#(S_u) < +\infty, \\ +\infty & \text{otherwise on } BV(I), \end{cases} \tag{19}$$

*where $\overline{\text{sub}}\,\theta$ is the greatest lower semicontinuous and subadditive function less than or equal to $\theta$ (see (20), (21) and Proposition 3 below).*

**Note.** We can substitute $|z|^2$ in the growth condition $f(z) \geq |z|^2 - c$ with any convex function growing more than linearly at infinity, with minor modifications in the proof.

*Remark 10.* Let $\overline{F}$ denote the relaxed functional of $F$ with respect to the $L^1$-topology on $BV(I)$. Then $\overline{F}(u) < +\infty$ only for functions $u \in SBV(I)$ with $\#(S_u) < +\infty$. Indeed, let $u \in BV(I)$ such that $\overline{F}(u) < +\infty$. By definition of relaxation, there exists a sequence $(u_h) \subset BV(I)$ such that $(u_h)$ converges to $u$ in $L^1$ and $\overline{F}(u) = \lim_{h \to +\infty} F(u_h) < +\infty$. By our assumptions on $f$ and $\theta$ we get $u_h \in SBV(I)\,, \#(S_{u_h}) \leq c\,, \|u_h\|_{BV} \leq c\,, \|(\dot{u_h})_a\|_{L^2} \leq c$. The compactness and the lower semicontinuity theorem on $SBV(I)$ (see Proposition 2) ensures that $u \in SBV(I)$ and $\#(S_u) < +\infty$.

*Remark 11.* By our assumptions on $f$ and $\theta$ (note $f$ has superlinear growth at infinity, and $\inf \theta > 0$) the relaxation process does not create interaction between the bulk energy and the jump-part energy density. Furthermore, by (16) and (17) we have $F(u) \geq |\dot{u}|(I) - c'$ : hence it is equivalent to consider sequences converging with respect to the $L^1(I)$-topology or with respect to the $BV - w^*$ topology.

*Remark 12.* A generalization to higher dimension of this result (under some further technical assumptions) can be found in [20], Prop. 7.1.

Before we start with the proof of Theorem 11 we state a compactness and lower semicontinuity result on $SBV(I)$. Moreover we give some attention to the notion of $\overline{\mathrm{sub}}\,\theta$ and state some of its properties (useful for the proof of Theorem 11). Their proofs and some examples will be given in the Appendix A.

**Proposition 2. (Compactness and lower semicontinuity theorem in $SBV(I)$)** *Let $(u_h)$ be a sequence in $SBV(I)$ such that*

$$\|u_h\|_{BV} \le c\,, \qquad \|(\dot u_h)_a\|_{L^2} \le c\,, \qquad \#(S_{u_h}) \le c\,.$$

*Then (possibly passing to a subsequence) there exists $u \in SBV(I)$ such that*

$$u_h \rightharpoonup u \text{ in } \; BV - w^*\,, \quad \text{and} \quad (\dot u_h)_a \rightharpoonup \dot u_a \text{ in } \; L^2(I)\,.$$

*Moreover, $\#(S_u) \le \liminf_{h\to+\infty} \#(S_{u_h})$. Furthermore, if $\theta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is lower semicontinuous and subadditive (i.e., $\theta$ satisfies 13), then*

$$\sum_{t\in S_u} \theta(u(t+), u(t-)) \le \liminf_{h\to+\infty} \sum_{t\in S_{u_h}} \theta(u_h(t+), u_h(t-))\,.$$

A proof of this result can be found in [21], Prop. 5.1. A more general $SBV$ compactness theorem can be found in [19] (see also [8] and references therein, and [3]).

**Definition:** The *subadditive envelope* $\mathrm{sub}\,\phi$ of $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the greatest subadditive function less than or equal to $\phi$.
It is easy to check that $\mathrm{sub}\,\phi \in \mathbb{R} \cup \{-\infty\}$ is given by the formula

$$\mathrm{sub}\,\phi(x,y) = \inf\Big\{\sum_{k=1}^{m} \phi(x_k, x_{k-1}) : x_0 = y, x_m = x, m = 1, 2, \dots\Big\}\,. \tag{20}$$

**Definition:** Given a function $\phi : \mathbb{R}\times\mathbb{R} \to [\,0, +\infty[$ we define the function $\overline{\mathrm{sub}}\,\phi : \mathbb{R}\times\mathbb{R} \to [\,0, +\infty[$ by setting

$$\overline{\mathrm{sub}}\,\phi(x,y) = \mathrm{sub}(sc^-\phi)(x,y)\,, \tag{21}$$

where $sc^-\phi$ is the lower semicontinuous envelope of $\phi$ (see Definition 4).

**Proposition 3.** *Let $\phi : \mathbb{R}\times\mathbb{R} \to [\,0, +\infty[$ be a function such that*

$$\inf \phi > 0\,, \qquad \phi(x,y) \ge |x-y| \quad \forall x, y \in \mathbb{R}\,.$$

*Then*

*(i)* $\inf(\overline{\mathrm{sub}}\,\phi) > 0$ *and* $\overline{\mathrm{sub}}\,\phi(x,y) \geq |x-y| \quad \forall x,y \in \mathbb{R}\,;$
*(ii)* $\overline{\mathrm{sub}}\,\phi$ *is the greatest lower semicontinuous and subadditive function less than or equal to* $\phi$.

*Proof of Theorem 11.* Without loss of generality let us fix here $I = \,]0,1[\,$. Let $H : BV(I) \to [\,0,+\infty\,]$ be the functional on the right-hand side of (19), i.e.,

$$H(u) = \begin{cases} \displaystyle\int_I f(\dot u_a)\,dt + \sum_{t\in S_u} \overline{\mathrm{sub}}\,\theta(u(t+),u(t-)) & \text{if } u \in SBV(I), \#(S_u) < +\infty, \\ +\infty & \text{otherwise on } BV(I). \end{cases}$$

We have to show that $\overline{F}(u)=$ (relaxed functional of $F$ with respect to the $L^1$-topology)$= H(u)$ for every $u \in BV(I)$.

**Step 1.** $H(u) \leq \overline{F}(u)$ for every $u \in BV(I)$. Since we have $H(u) \leq F(u)$ for all $u \in BV(I)$, it is enough to show that $H$ is $L^1$-lower semicontinuous on $BV(I)$. Let $u \in BV(I)$ and let $(u_h)$ converge to $u$ in $L^1(I)$ such that $\lim\limits_{h\to+\infty} H(u_h) < +\infty$. Hence we can suppose $(u_h) \subset SBV(I)$. Since $f(z) \geq |z|^2 - c$, and $\overline{\mathrm{sub}}\,\theta(x,y) \geq |x-y|$ we get then

$$\|(\dot u_h)_a\|_{L^2} \leq c \qquad \text{and} \qquad \|u_h\|_{BV} \leq c\,.$$

Let us remark that $\inf \overline{\mathrm{sub}}\,\theta > 0$, and hence we have also

$$\#(S_{u_h} \cap I) \leq \frac{H(u_h)}{\inf \overline{\mathrm{sub}}\,\theta} \leq c < +\infty\,.$$

By applying Proposition 2 we get $u \in SBV(I)$, $\#(S_u \cap I) < +\infty$ and

$$\sum_{t\in S_u} \overline{\mathrm{sub}}\,\theta(u(t+),u(t-)) \leq \liminf_{h\to+\infty} \sum_{t\in S_{u_h}} \overline{\mathrm{sub}}\,\theta(u_h(t+),u_h(t-))\,. \tag{22}$$

Furthermore we can suppose $(\dot u_h)_a \rightharpoonup (\dot u)_a$ in $L^2(I)$. Our assumptions on $f$ then imply (see [33], Ex. 1.22, Ex. 1.23)

$$\int_I f(\dot u_a)\,dt \leq \liminf_{h\to+\infty} \int_I f((\dot u_h)_a)\,dt\,. \tag{23}$$

By (22) and (23) we get $H(u) \leq \lim\limits_{h\to+\infty} H(u_h)$.

**Step 2.** $H(u) \geq \overline{F}(u)$ for every $u \in BV(I)$. By Remark 10 it is enough to prove the above inequality for $u \in SBV(I)$ with $\#(S_u) < +\infty$. We have now to show that for every $u \in SBV(I)$ with $\#(S_u) < +\infty$ we can build up a recovery sequence $(u_h)$ such that

$$u_h \to u \quad \text{in} \quad L^1(I)$$

and

$$H(u) \geq \liminf_{h\to+\infty} F(u_h)\,.$$

We can study the case of a single jump without losing in generality. We can suppose $u \in SBV(I)$ and

$$\dot u_s = (u(t_0+) - u(t_0-))\delta_{t_0}$$

for some $t_0 \in I$. By the definition of $\overline{\mathrm{sub}}\,\theta$, for every $h \in \mathbb{N}$ there exist real numbers $a_0^h, a_1^h, \ldots, a_{N_h}^h$ and $b_0^h, b_1^h, \ldots, b_{N_h}^h$ such that we have

$$|b_0^h - u(t_0-)| \leq \frac{1}{h^2}\,, \quad |a_{N_h}^h - u(t_0+)| \leq \frac{1}{h^2}\,, \quad |b_j^h - a_{j-1}^h| \leq \frac{1}{h^2} \tag{24}$$

for every $j = 1, \ldots, N_h$ and

$$\overline{\mathrm{sub}}\,\theta(u(t_0+), u(t_0-)) + \frac{1}{h} \;\geq\; \sum_{j=0}^{N_h} \theta(a_j^h, b_j^h)\,. \tag{25}$$

By our assumptions (17) on $\theta$, the inequality (25) implies that

$$N_h \leq c$$

and

$$\sum_{j=0}^{N_h} |a_j^h - b_j^h| \leq c\,.$$

We can suppose that $N_h = N$ independent of $h \in \mathbb{N}$. Let us fix $M \in \mathbb{N}$ such that

$$\left]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}\right[ \;\subset\; I\,.$$

For every $h \geq MN$ (so that $\frac{1}{M} > \frac{N}{h}$) we define $u_h \in SBV(I)$ by setting

$$u_h(0+) = u(0+) \quad \text{and} \quad \dot u_h = w_h(t)\,dt + \sum_{j=0}^{N} (a_j^h - b_j^h)\delta_{(t_0 - \frac{N-j}{h})}\,,$$

where

$$w_h(t) = \begin{cases} \dot u_a(t) & \text{if} \quad t < t_0 - \dfrac{1}{M} \\ \dot u_a(t) + \frac{Mh}{h-MN}(b_0^h - u(t_0-) & \text{if} \quad t_0 - \dfrac{1}{M} \leq t < t_0 - \dfrac{N}{h} \\ (b_j^h - a_{j-1}^h)h & \text{if} \quad t_0 - \dfrac{N-j+1}{h} \leq t \text{ and} \\ & \quad t < t_0 - \dfrac{N-j}{h} \quad 1 \leq j \leq N \\ \dot u_a(t) + M(u(t_0+) - a_N^h) & \text{if} \quad t_0 \leq t < t_0 + \dfrac{1}{M} \\ \dot u_a(t) & \text{if} \quad t \geq t_0 + \dfrac{1}{M} \end{cases}$$

(obviously $w_h \in L^1(I \setminus S_{u_h})$ and $|(\dot{u}_h)_s| = \sum_{j=0}^{N} |a_j^h - b_j^h| < +\infty$). Remark that $u_h(t) = u(t)$ outside $]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[$ for every $h$. Moreover, $t_0 - \frac{N}{h} \to t_0$ as $h$ tends to $+\infty$ and $(w_h)$ converges to $\dot{u}_a$ on $]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[$. We then have

$$u_h \;\to\; u \qquad \text{in } \; L^1(I)$$

and

$$\begin{aligned} F(u_h) = & \int_{I \setminus [t_0 - \frac{1}{M}, t_0 + \frac{1}{M}]} f((\dot{u}_h)_a)\, dt + \int_{]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[} f((\dot{u}_h)_a)\, dt + \\ & + \sum_{t \in S_{u_h}} \theta(u_h(t+), u_h(t-)) \\ = & \int_{I \setminus [t_0 - \frac{1}{M}, t_0 + \frac{1}{M}]} f((\dot{u})_a)\, dt + \int_{]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[} f(w_h)\, dt + \sum_{j=0}^{N} \theta(a_j^h, b_j^h) \\ \le & \int_{I \setminus [t_0 - \frac{1}{M}, t_0 + \frac{1}{M}]} f((\dot{u})_a)\, dt + \int_{]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[} f(w_h)\, dt + \\ & + \overline{\text{sub}}\, \theta(u(t_0+), u(t_0-)) + \frac{1}{h}\,. \end{aligned} \tag{26}$$

Now

$$\begin{aligned} \int_{]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[} f(w_h)\, dt = & \int_{t_0 - \frac{1}{M}}^{t_0 - \frac{N}{h}} f(\dot{u}_a + \varepsilon(h))\, dt + \int_{t_0 - \frac{N}{h}}^{t_0} f(\varepsilon'(h))\, dt + \\ & + \int_{t_0}^{t_0 + \frac{1}{M}} f(\dot{u}_a + \varepsilon''(h))\, dt\,, \end{aligned}$$

where $\varepsilon(h)$, $\varepsilon'(h)$ and $\varepsilon''(h)$ are functions which vanish as $h$ tends to $+\infty$ (recall (24)); by continuity we get

$$\lim_{h \to +\infty} \int_{]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[} f(w_h)\, dt = \int_{]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[} f(\dot{u}_a)\, dt\,.$$

Hence, by passing to the limit in (26) we get

$$\limsup_{h \to +\infty} F(u_h) \le \int_I f(\dot{u}_a)\, dt + \overline{\text{sub}}\, \theta(u(t_0+), u(t_0-)) = H(u)\,.$$

It is clear that in the same way we can treat the general case of more then one jump. □

*Remark 13.* Let us note that in the previous proof Step 1 (that is $H \le \overline{F}$) has been proved using a lower semicontinuity result in $SBV(I)$ (see Proposition

2) whereas Step 2 (that is $H \geq \overline{F}$) has been shown building up a "recovering" sequence. As we will see, this scheme will be used more in general as far as the first step is concerned, while for Step 2 we have in general (see Section 7 and comments in Section 8) to proceed in a completely different, more technical manner.

### 5.2 A sufficient condition for l.s.c. for functionals defined on $SBV(I)$. Relaxation in $BV(I)$

Let us state the following semicontinuity theorem for a functional $F$ like (12) where the assumption on $\theta$ to satisfy $\inf \theta > 0$ is dropped, but a condition of superlinearity at the origin is added (see [5]; see also [19]). Subsequently we will prove a relaxation result when possibly $\inf \theta = 0$ and no superlinearity condition at the origin is assumed.

**Theorem 12 (Lower semicontinuity - superlinear growth conditions for the bulk and jump-part energy densities).** *Let $f : \mathbb{R} \to [0, +\infty]$ be convex and*

$$\lim_{|z| \to +\infty} \frac{f(z)}{|z|} = +\infty \qquad \textit{(i.e., } f \textit{ is superlinear at } \infty\textit{).} \tag{27}$$

*Let $\theta : \mathbb{R} \to [0, +\infty]$ be lower semicontinuous and subadditive (see (15) and*

$$\lim_{|z| \to 0} \frac{\theta(z)}{|z|} = +\infty \qquad \textit{(i.e., } \theta \textit{ is superlinear at } 0\textit{).} \tag{28}$$

*Let $I \subset \mathbb{R}$ be a bounded open interval. Then the functional $F : BV(I) \to [0, +\infty]$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_I f(\dot{u}_a)\, dt + \sum_{t \in S_u} \theta(u(t+) - u(t-)) & \textit{if } u \in SBV(I), \\ +\infty & \textit{if } u \in BV(I) \setminus SBV(I) \end{cases} \tag{29}$$

*is $BV - w^*$ lower semicontinuous on $BV(I)$.*

*Remark 14.* Let us note that the hypotheses on $f$ and $\theta$ imply that $f(z) \geq |z| - c$, and $\theta(z) \geq |z| - c'$ for some suitable non-negative constants $c$ and $c'$. Since in general $c' \neq 0$, we get

$$F(u) \geq |\dot{u}|(I) - c' \#(S_u \cap I) - c''\,. \tag{30}$$

For functionals satisfying (30) it is in general not equivalent to consider sequences converging with respect to the $L^1$-topology or with respect to the $BV - w^*$ topology ($u \in BV(I)$ or $u \in SBV(I)$ does not imply $\#(S_u \cap I) < +\infty$!).

*Remark 15.* We can take for example $\theta(z) = \sqrt{|z|}$, $z \in \mathbb{R}$; more in general, every function $\theta(z) = \varphi(|z|)$, $z \in \mathbb{R}$ with $\varphi : [\,0, +\infty[ \to [\,0, +\infty[$ concave, non-decreasing and $\lim_{z \to 0^+} \dfrac{\varphi(z)}{z} = +\infty$ satisfies the hypotheses of Theorem 12 (see Remark 9 Example (iii)). In particular, $\theta(z) = c \in \mathbb{R}$ can be considered. In the higher dimensional case this result can be found in [5]. A lower semicontinuity result with respect to the $L^1$-topology for functionals of the form (29) with $\theta = \theta(t, z) = 1 + a(t)|z|$, and $a : \overline{I} \to \mathbb{R}$ strictly positive and continuous, can be found in [30].

**Note:** The function $\theta(x, y) = |x - y|$ does not fit the assumptions of Theorem 10 (indeed $\inf \theta = 0$) as well as the function $\theta(z) = |z|$ does not satisfies hypothesis (28) of Theorem 12.

The following theorem deals then with the "linear" case, and no superlinearity conditions for $f$ are required.

**Theorem 13 (Relaxation in $BV(I)$ - "linear" growth case: interaction between bulk and jump-part energy densities).** *Let $f : \mathbb{R} \to [\,0, +\infty[$ be convex such that $f(0) = 0$, and let $I \subset \mathbb{R}$ be a bounded open interval. Let $F : BV(I) \to [\,0, +\infty\,]$ be the functional defined by*

$$F(u) = \begin{cases} \displaystyle\int_I f(\dot{u}_a)\, dt + \sum_{t \in S_u} |u(t+) - u(t-)| & \text{if } u \in SBV(I), \\ +\infty & \text{otherwise on } BV(I). \end{cases} \tag{31}$$

*The relaxed functional with respect to the $L^1(I)$-topology, denoted by $\overline{F}$, can be written as*

$$\overline{F}(u) = \int_I \overline{f}(\dot{u}_a)\, dt + \int_I \overline{f}^{\infty}\Big(\frac{\dot{u}_s}{|\dot{u}_s|}\Big)\, |\dot{u}_s| \qquad u \in BV(I)\,, \tag{32}$$

*where $\overline{f}(z) = (f(z) \wedge |z|)^{**}$ for every $z \in \mathbb{R}$ and $\overline{f}^{\infty}(z)$ is its recession function.*

*Remark 16.* Note that the right-hand side of (32) is defined on the whole $BV(I)$. Recall that $\dfrac{\dot{u}_s}{|\dot{u}_s|}$ denotes the Radon-Nikodým derivative of $\dot{u}_s$ with respect to $|\dot{u}_s|$ and that we use the notation $\displaystyle\int_I h(x)\, |\dot{u}_s|$ instead of $\displaystyle\int_I h(x)\, d|\dot{u}_s|$.

*Remark 17.* Let us point out that the energy densities which appear in the integral representation (32) take into account simultaneously both the bulk energy density $f(z)$ as well as the jump-part energy density $\theta(z) = |z|$.

If we take now $f$ and $\theta$ as in Theorem 13 we have immediately $\theta^0(z) = \theta(z)$ (see (9). Moreover (see (83))

$$\overline{f}(z) = (f(z) \wedge \theta^0(z))^{**}$$

and

$$\overline{f}^{\infty}(z) = \overline{\text{sub}}\,(f^{\infty}(z) \wedge \theta(z))\,.$$

We will notice briefly the competition between the functions $f$ and $\theta$ in order to find a "right energy balance": the function $f$ will be preferred with respect to the function $\theta$ if $\theta$ has a fast growth for small $z$ (that is, regularity will be preferred instead of small jumps); on the other hand, if the function $f$ grows more than linearly at infinity the function $\theta$ should be selected (that is, fast growth of the functions will be replaced by jumps).

*Example 1.* Take $f(z) = z^2$ ; we get easily that

$$\overline{f}(z) = \begin{cases} z^2 & \text{if } |z| \le \frac{1}{2} \\ |z| - \frac{1}{4} & \text{otherwise,} \end{cases}$$

and

$$\overline{f}^{\infty}(z) = \lim_{t\to+\infty} \frac{\overline{f}(tz)}{t} = |z|\,.$$

In order to prove Theorem 13 we shall need the following results (stated here for our convenience in dimension 1) about relaxation in $BV$ and $W^{1,1}$, due to Goffman and Serrin, and Buttazzo and Dal Maso, respectively.

**Theorem 14.** ([39]) *Let $V : \mathbb{R} \to [\,0, +\infty[$ be a convex function such that*

$$0 \le V(z) \le c(1 + |z|)\,.$$

*Let $E : BV(I) \to [\,0, +\infty\,]$ be the functional defined by*

$$E(u) = \begin{cases} \displaystyle\int_I V(\dot{u}_a)\,dt & \text{if } u \in W^{1,1}(I), \\ +\infty & \text{if } u \in BV(I) \setminus W^{1,1}(I). \end{cases}$$

*Then the relaxed functional of $E$ with respect to the $L^1$-topology is given by*

$$\overline{E}(u) = \int_I V(\dot{u}_a)\,dt + \int_I V^{\infty}\big(\frac{\dot{u}_s}{|\dot{u}_s|}\big)\,|\dot{u}_s| \tag{33}$$

*for every $u \in BV(I)$.*

**Theorem 15.** ([28]) *Let $V : \mathbb{R} \to [\,0, +\infty[$ be a Borel function such that*

$$c_1|z| - c_2 \le V(z) \le c(1 + |z|)\,.$$

*Let $E : W^{1,1}(I) \to [\,0, +\infty\,]$ be the functional defined by*

$$E(u) = \begin{cases} \displaystyle\int_I V(\dot{u}_a)\,dt & \text{if } u \in C^1(I) \cap W^{1,1}(I), \\ +\infty & \text{otherwise on } W^{1,1}(I). \end{cases}$$

*Then the relaxed functional of $E$ with respect to the $L^1$-topology is given by*

$$\overline{E}(u) = \int_I V^{**}(\dot{u}_a)\,dt \tag{34}$$

*for every $u \in W^{1,1}(I)$.*

*Proof of Theorem 13* (see [30], Theorem 3.1, and [21], Section 4). Let $H : BV(I) \to [\,0, +\infty\,]$ be the functional on the right-hand side of (32), i.e.,

$$H(u) = \int_I \overline{f}(\dot{u}_a)\,dt + \int_I \overline{f}^{\infty}\big(\frac{\dot{u}_s}{|\dot{u}_s|}\big)\,|\dot{u}_s|\,.$$

We have to show that $\overline{F}$(u)= (relaxed functional of $F$ with respect to the $L^1$-topology)=$H(u)$ for every $u \in BV(I)$.

**Step 1.** $H(u) \le \overline{F}(u)$ for every $u \in BV(I)$. Let us note that $\overline{f}(z) = (f(z) \wedge |z|)^{**} \le |z|$ for all $z \in \mathbb{R}$. Hence we can apply Theorem 14 with $V(z) = \overline{f}(z)$; hence $H(u)$ is lower semicontinuous with respect to the $L^1$-topology on $BV(I)$ (by definition of relaxation). Now it is enough to show that $H(u) \le F(u)$ for every $u \in BV(I)$: clearly

$$\overline{f}(z) = (f(z) \wedge |z|)^{**} \le f(z) \qquad \forall z \in \mathbb{R}\,.$$

Furthermore, $\overline{f}^{\infty}(z) = \lim_{t\to+\infty} \dfrac{\overline{f}(tz)}{t} \le \lim_{t\to+\infty} \dfrac{|tz|}{t} = |z|$ and hence

$$\int_I \overline{f}^{\infty}\big(\frac{\dot{u}_s}{|\dot{u}_s|}\big)\,|\dot{u}_s| \le \int_I \Big|\frac{\dot{u}_s}{|\dot{u}_s|}\Big|\,|\dot{u}_s| = \int_I |\dot{u}_s| = |\dot{u}_s|(I)\,.$$

Since

$$|\dot{u}_s|(I) \le \begin{cases} \displaystyle\sum_{t\in S_u} |u(t+) - u(t-)| & \text{if } u \in SBV(I), \\ +\infty & \text{on } BV(I) \setminus SBV(I) \end{cases}$$

we may conclude that $H(u) \le F(u)$ for every $u \in BV(I)$. By definition of relaxation we get then $H \le \overline{F}$ on $BV(I)$.

**Step 2.** $H(u) \ge \overline{F}(u)$ for every $u \in BV(I)$. It is enough to show that

$$\overline{F}(u) \le \begin{cases} \displaystyle\int_I \widetilde{f}(\dot{u}_a)\,dt & \text{if } u \in C^1(I) \cap W^{1,1}(I), \\ +\infty & \text{otherwise on } W^{1,1}(I), \end{cases} \tag{35}$$

where $\widetilde{f}(z) = f(z) \wedge |z|$. Indeed, by Theorem 15 with $V(z) = \widetilde{f}(z)$ (recall that by our assumptions we have $c|z| - c' \le f(z)$ for suitable positive constants $c$ and $c'$) we get

$$\overline{F}(u) \le \int_I \overline{f}(\dot{u}_a)\,dt \qquad \text{for every } u \in W^{1,1}(I). \tag{36}$$

Since clearly

$$\overline{F}(u) \le \begin{cases} \displaystyle\int_I \overline{f}(\dot{u}_a)\,dt & \text{if } u \in W^{1,1}(I), \\ +\infty & \text{if } u \in BV(I) \setminus W^{1,1}(I), \end{cases}$$

by applying again Theorem 14 with $V(z) = \overline{f}(z)$ we get finally

$$\overline{F}(u) \le H(u) \qquad \forall u \in BV(I)\,.$$

To conclude the proof of Step 2 it remains therefore to prove (35).
Let $u \in C^1(I) \cap W^{1,1}(I)$ : let us set

$$I_u = \{t \in I : f(\dot{u}_a(t)) > |\dot{u}_a(t)|\}$$

(recall that $\dot{u}_a$ coincides with the classical derivative of $u$ in $C^1$). We approximate $u$ on $I_u$ with piecewise constant functions, thus replacing quick growth of $u$ by a sequence of jumps. The set $I_u$ is open, hence, considering its connected components, it can be split in an at most countable union of open intervals of $I$:

$$I_u = \bigcup_{i=1}^{\infty} I_u^i\,.$$

We construct a sequence $(u_k)_k$ as follows: fixed $h \in \mathbb{N}$ we subdivide every interval $I_u^i$ in a finite number of contiguous subintervals of length less than $1/h$ :

$$I_u^i = \bigcup_{m=1}^{n_{i,h}} I_m^{i,h} \qquad |I_m^{i,h}| < \frac{1}{h}\,.$$

Set $a_m^{i,h} = \inf I_m^{i,h}$ for $m \le n_{i,h}$ and $a_{n_{i,h}}^{i,h} = \sup I_u^i$. Let us consider now for every $k \in \mathbb{N}$ the set $\bigcup_{i=1}^k I_u^i$ and let us set

$$u_k^h(t) = \begin{cases} \displaystyle\sum_{i=1}^{k} \sum_{m=1}^{n_{i,h}} u(a_m^{i,h}) \mathbf{1}_{I_m^{i,h}}(t) & \text{if } t \in \displaystyle\bigcup_{i=1}^{k} I_u^i, \\ u(t) & \text{if } t \in I \setminus \displaystyle\bigcup_{i=1}^{k} I_u^i. \end{cases}$$

Let us take now $h = h(k)$ such that $h(k) \to +\infty$ as $k \to +\infty$ and

$$\|u_k^h - u\|_{L^1(I)} < \frac{1}{k} \qquad \text{(integral sum of Cauchy)}$$

and set, by definition, $u_k = u_k^{h(k)}$. Clearly $u_k \in SBV(I)$ and we get

$$F(u_k) = \int_I f((\dot{u}_k)_a)\,dt + \sum_{i=1}^{k} \sum_{m=1}^{n_{i,h(k)}} |u(a_{m+1}^{i,h(k)}) - u(a_m^{i,h(k)})|$$

$$\leq \int_{I \setminus \bigcup_{i=1}^{k} I_u^i} f(\dot{u}_a)\,dt + \sum_{i=1}^{k} \sum_{m=1}^{n_{i,h(k)}} \int_{a_m^{i,h(k)}}^{a_{m+1}^{i,h(k)}} |\dot{u}_a|\,dt$$

$$\leq \int_{I \setminus \bigcup_{i=1}^{k} I_u^i} f(\dot{u}_a)\,dt + \int_{I_u} |\dot{u}_a|\,dt\,.$$

Since $\int_{I \setminus I_u^1} f(\dot{u}_a)\,dt < +\infty$, passing to the limit as $k \to +\infty$ (by the continuity from above) we get

$$\limsup_{k \to +\infty} F(u_k) \leq \int_{I \setminus I_u} f(\dot{u}_a(t))\,dt + \int_{I_u} |\dot{u}_a(t)|\,dt \leq \int_I \widetilde{f}(\dot{u}_a(t))\,dt\,;$$

hence

$$\overline{F}(u) \leq \begin{cases} \int_I \widetilde{f}(\dot{u}_a(t))\,dt & \text{if } u \in C^1(I) \cap W^{1,1}(I), \\ +\infty & \text{otherwise on } W^{1,1}(I), \end{cases}$$

that is, (35) holds. □

*Remark 18.* With the same arguments of the proof of Theorem 13 we can prove that if we substitute in the definition of the functional $F$ the term $\sum_{t \in S_u} |u(t+) - u(t-)|$ with $\gamma \sum_{t \in S_u} |u(t+) - u(t-)|$, where $\gamma$ is a positive constant, then the relaxation of $F$ has the same expression with $\overline{f}(z) = (f(z) \wedge \gamma|z|)^{**}$.

**Corollary 1.** *Let $f$ be as in Theorem 13. Let $F_0 : BV(I) \to [0, +\infty]$ be the functional defined by*

$$F_0(u) = \begin{cases} \int_I f(\dot{u}_a)\,dt + \sum_{t \in S_u} |u(t+) - u(t-)| & \textit{if } u \in SBV(I), \#(S_u \cap I) < +\infty, \\ +\infty & \textit{otherwise on } BV(I). \end{cases} \tag{37}$$

*Then the relaxed functional $\overline{F_0}$ of $F_0$ with respect to the $L^1$-topology coincides with $\overline{F}$ given in (32), i.e., we can take minimizing sequences for $\overline{F}$ with a finite number of jumps.*

*Proof.* Let $(u_h) \subset SBV(I)$ such that $u_h \to u$ in $L^1(I)$ and $\overline{F}(u) = \lim_{h \to +\infty} F(u_h)$, with $\overline{F}$ the relaxed functional of $F$ with respect to the $L^1$-topology. We can write the measure

$$\dot{u}_h = (\dot{u}_h)_a \, dt + \sum_{k=1}^{\infty} a_k^h \delta_{t_k^h}$$

with $S^h = \sum_{k=1}^{\infty} |a_k^h| < +\infty$. Fixed $h \in \mathbb{N}$, let $N_h \in \mathbb{N}$ be such that

$$\Big|\sum_{k=1}^{N_h} |a_k^h| - S^h\Big| < \frac{1}{h}\,.$$

Then there exists a function $v_h \in SBV(I)$ such that

$$\dot{v}_h = (\dot{u}_h)_a \, dt + \sum_{k=1}^{N_h} a_k^h \delta_{t_k^h} \qquad \text{and} \qquad \|u_h - v_h\|_\infty < \tfrac{1}{h}.$$

We have $v_h \to u$ in $L^1(I)$ and $\lim_{h\to+\infty} F(v_h) = \lim_{h\to+\infty} F(u_h) = \overline{F}(u)$. □

*Remark 19.*

(i) A relaxation result with respect to the $L^1(I)$-topology for functionals of the form

$$F(u) = \int_I f(\dot{u}_a)\, dt + \#(S_u) + |\dot{u}_s|(I) \qquad \text{on } BV(I),$$

$$G(u) = \int_I f(\dot{u}_a)\, dt + \#(S_u) + \int_I a(t)\, |\dot{u}_s| \qquad \text{on } BV(I)$$

can be found in [30].

(ii) Other lower semicontinuity and relaxation results on $BV(I;\mathbb{R}^m)$, $m \geq 1$ (for the $BV - w^*$ convergence) for functionals of the form (29) can be found in [15], [16] (see Section 8). Let us notice that they treat also the case where $f = f(t,z)$ and $\theta = \theta(t,z)$.

## 6 Lower semicontinuity and relaxation in higher dimension

The lower semicontinuity and relaxation theorems of the previous section can be generalized in many ways to the higher dimensional case. Let us state first a simple result of lower semicontinuity for functionals having superlinear growth of the bulk and surface energy densities (see [19], Cor. 2.15, [5] ). Let $n \geq 1$.

**Theorem 16 (Lower semicontinuity for functionals defined on $SBV(\Omega)$ - superlinear growth of the bulk and surface energy densities).** *Let $f : \mathbb{R}^n \to [\,0, +\infty[$ be convex and*

$$\lim_{|z|\to+\infty} \frac{f(z)}{|z|} = +\infty \qquad \textit{(i.e., } f \textit{ is superlinear at } \infty\textit{)}. \tag{38}$$

*Let $\theta : \mathbb{R} \times \mathbb{R} \to [\,0, +\infty[$ be a lower semicontinuous, symmetric subadditive function, i.e., $\theta$ is lower semicontinuous and*

$$\theta(z,w) = \theta(w,z) \le \theta(w,y) + \theta(y,z) \qquad \forall w,y,z \in \mathbb{R}\,. \tag{39}$$

*Let $\psi : [\,0,+\infty[ \to [\,0,+\infty\,]$ be a function such that $\theta(z,w) \ge \psi(|z-w|)$ for every $z, w \in \mathbb{R}$ and*

$$\lim_{|z|\to 0^+} \frac{\psi(z)}{z} = +\infty \qquad \textit{(implies } \theta \textit{ is superlinear at } 0\textit{)}. \tag{40}$$

*Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Then the functional*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,dx + \int_{S_u\cap\Omega} \theta(u^+,u^-)\,dH^{n-1} & \textit{if } u \in SBV(\Omega), \\ +\infty & \textit{on } BV(\Omega)\setminus SBV(\Omega) \end{cases} \tag{41}$$

*is $BV-w^*$ lower semicontinuous on $BV(\Omega)$. If, in addition, $\theta(z,w) \ge c|z-w|$ for every $z\,, w \in \mathbb{R}$, then $F$ is lower semicontinuous on $BV(\Omega)$ with respect to the $L^1$-topology.*

*Example 2.* (i) We can take $\theta(z,w) = \sqrt{|z-w|}$ (in this case $\psi(s) = \sqrt{s}$ for $s \ge 0$) and the functional in (41) becomes of the form

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,dx + \int_{S_u\cap\Omega} \sqrt{|u^+ - u^-|}\,dH^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise on } BV(\Omega). \end{cases}$$

(ii) Also $\theta(z,w) \equiv 1$ (or any other constant) can be considered in Theorem 16. In this case we have

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,dx + H^{n-1}(S_u\cap\Omega) & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise on } BV(\Omega). \end{cases}$$

In this context let us emphasize that lower semicontinuity results for functionals modeled on the image segmentation functionals introduced by Mumford and Shah

$$F(u) = \int_\Omega |\nabla u|^2\,dx + c_1 \int_\Omega |u-g|^2\,dx + c_2 H^{n-1}(S_u\cap\Omega)$$

for $u \in SBV(\Omega)$ with $H^{n-1}(S_u\cap\Omega) < +\infty$ and $\nabla u \in L^2(\Omega;\mathbb{R}^n)$, are due to [4],[5] (in this case, the "grey function" $g$, with $0 \le g \le 1$, represents the input picture, and $S_u$ is expected to detect the relevant contours of the objects in the picture). Let us point out that the term $c_1 \int_\Omega |u-g|^2\,dx$ is trivially lower semicontinuous by the Fatou lemma, and can therefore be neglected in the study of the semicontinuity of $F$.

(iii) In the context of Example ii) Braides and Chiadò Piat ([20]) proved some relaxation results for functionals of the form

$$G(u) = \int_\Omega f(\nabla u)\,dx + c_1 \int_\Omega |u-g|^p\,dx + c_2 \int_{S_u \cap \Omega} \theta(u^+, u^-)\,dH^{n-1}\,,$$

with $u \in SBV(\Omega)$, $H^{n-1}(S_u \cap \Omega) < +\infty$ and $\nabla u \in L^p(\Omega;\mathbb{R}^n)$. In addition to some regularity assumptions on $f$ and $\theta$ (which is also increasing) they assume that $f$ has $p$-growth, i.e., $\alpha|\xi|^p \le f(\xi) \le \beta(1+|\xi|^p)$, for some $p > 1\,, \alpha\,, \beta > 0$ and $\alpha \le \theta(x,y) \le \beta(1+|x-y|)$ for all $x, y \in \mathbb{R}$.

Let us now present a relaxation result for functionals which cannot be handled by Theorem 16 since in particular a condition like (40) fails. This result is the $n$-dimensional version of Theorem 13. In this case, the lower bound for the relaxed functional will be obtained as in the 1-dimensional case while for the upper bound we must use fine properties of the $BV$ functions (in particular, we will use the co-area formula for $BV$ functions to construct the "recovery"-sequence (see [21], Theorem 3.1).

**Theorem 17 (Relaxation in $BV(\Omega)$ - "linear" growth case: interaction between bulk and surface energy densities).** *Let $f : \mathbb{R}^n \to [\,0, +\infty\,]$ be a lower semicontinuous and convex function, and $f(0) = 0$. Furthermore, assume that the set*

$$K = \{z \in \mathbb{R}^n : f(z) \le |z|\} \tag{42}$$

*is bounded. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $F : BV(\Omega) \to [\,0, +\infty\,]$ be the functional defined by*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,dx + \int_{S_u \cap \Omega} |u^+ - u^-|\,dH^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise on } BV(\Omega). \end{cases} \tag{43}$$

*The relaxed functional with respect to the $L^1(\Omega)$-topology, denoted by $\overline{F}$, can be written for $u \in BV(\Omega)$ as*

$$\begin{aligned} \overline{F}(u) &= \int_\Omega \overline{f}(\nabla u)\,dx + \int_\Omega \overline{f}^\infty\Big(\frac{D_s u}{|D_s u|}\Big)\,|D_s u| \\ &= \int_\Omega \overline{f}(\nabla u)\,dx + \int_\Omega |D_s u|\,, \end{aligned} \tag{44}$$

*where $\overline{f}(z) = (f(z) \wedge |z|)^{**}$ for every $z \in \mathbb{R}^n$ and $\overline{f}^\infty$ is its recession function.*

Remember that $\dfrac{D_s u}{|D_s u|}$ is the Radon-Nikodým derivative of the measure $D_s u$ with respect to its total variation, and that we write $\displaystyle\int_\Omega h(x)\,|D_s u|$ instead of $\displaystyle\int_\Omega h(x)\,d|D_s u|$.

*Remark 20.* The above theorem can be extended to the vector-valued case under the same hypotheses on $f$. Moreover, this result can be obtained easily as a corollary of a more general relaxation result proved by Braides and Coscia ([22]) and treated in Section 7 (see Corollary 2).

*Remark 21.* Let us remark that, once $f \neq +\infty$, it is not restrictive to suppose that $f(0) = 0$. In fact, otherwise we could consider the function $f_1(z) = f(z + z_0) - f(z_0)$, where $f(z_0) = \min f$.

Notice also that we have $\overline{f}^\infty(z) = |z|$ : indeed, it is easy to see that there exists $R > 0$ such that

$$K \subset B_R(0) \qquad \text{and} \qquad |z| - R \leq \overline{f}(z) \leq |z|\,. \tag{45}$$

By (45) we get then immediately $\overline{f}^\infty(z) = \displaystyle\lim_{t\to+\infty} \frac{\overline{f}(tz)}{t} = |z|$. Since $\overline{f}^\infty(z) = |z|$, we get immediately the last equality in (44) since $\left|\dfrac{D_s u}{|D_s u|}\right| = 1$ $|D_s u|$–a.e. on $\Omega$.

*Remark 22.* Let us underline that assumption (42) implies that the $L^1(\Omega)$-relaxed functional of $F$ coincides with the $BV - w^*$-relaxed functional of $F$ (if $\Omega$ has a compact Lipschitz boundary). Indeed, let us denote by $\overline{F}_{BV-w^*}$ the relaxed functional of $F$ with respect to the $BV - w^*$ topology. Clearly we have $\overline{F}_{BV-w^*} \geq \overline{F}$. On the other hand, let $(u_h) \subset BV(\Omega)$ such that $(u_h)$ converges to $u$ in $L^1(\Omega)$ and $\overline{F}(u) = \displaystyle\lim_{h\to+\infty} F(u_h) < +\infty$. For every $h \in \mathbb{N}$ consider $\Omega_h = \{x \in \Omega : \nabla u_h(x) \in K\} = \{x \in \Omega : f(\nabla u_h(x)) \leq |\nabla u_h(x)|\}$ and set $\Omega_h' = \Omega \setminus \Omega_h$. We get then

$$\begin{aligned} |Du_h| &= \int_{\Omega_h} |\nabla u_h(x)|\,dx + \int_{\Omega_h'} |\nabla u_h(x)|\,dx + \int_{S_u\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} \\ &\leq R|\Omega_h| + \int_{\Omega_h'} f(\nabla u_h(x))\,dx + \int_{S_u\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} \\ &\leq R|\Omega| + c\,. \end{aligned}$$

(Remember that for every $x \in \Omega_h'$ we have $|\nabla u_h(x)| \leq R$; moreover, we have supposed that $\lim_h F(u_h)$ exists finite). Passing possibly to a subsequence we obtain that $u_h \rightharpoonup u$ $BV - w^*$ and therefore $\overline{F}_{BV-w^*}(u) \leq \liminf_h F(u_h) = \overline{F}(u)$.

Let us state now in dimension $n > 1$ the relaxation theorems by Goffman and Serrin and by Buttazzo and Dal Maso, respectively, used in Section 5 for $n = 1$.

**Theorem 18.** ([39]) *Let* $V : \mathbb{R}^n \to [\,0, +\infty[$ *be a convex function such that*

$$0 \le V(z) \le c(1 + |z|)\,.$$

*Let* $E : BV(\Omega) \to [\,0, +\infty\,]$ *be the functional defined by*

$$E(u) = \begin{cases} \displaystyle\int_\Omega V(\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{if } u \in BV(\Omega) \setminus W^{1,1}(\Omega). \end{cases}$$

*Then the relaxation of* $E$ *with respect to the* $L^1$*-topology is given for any* $u \in BV(\Omega)$ *by*

$$\overline{E}(u) = \int_\Omega V(\nabla u)\,dx + \int_\Omega V^\infty(\frac{D_s u}{|D_s u|})\,|D_s u|\,. \tag{46}$$

**Theorem 19.** ([28]) *Let* $V : \mathbb{R}^n \to [\,0, +\infty[$ *be a Borel function such that*

$$c_1|z| - c_2 \le V(z) \le c(1 + |z|)\,.$$

*Let* $E : W^{1,1}(\Omega) \to [\,0, +\infty\,]$ *be the functional defined by*

$$E(u) = \begin{cases} \displaystyle\int_\Omega V(\nabla u)\,dx & \text{if } u \in C^1(\Omega) \cap W^{1,1}(\Omega), \\ +\infty & \text{if } u \in W^{1,1}(\Omega) \setminus C^1(\Omega). \end{cases}$$

*Then the relaxed functional of* $E$ *with respect to the* $L^1$*-topology is given by*

$$\overline{E}(u) = \int_\Omega V^{**}(\nabla u)\,dx \tag{47}$$

*for every* $u \in W^{1,1}(\Omega)$.

*Proof of Theorem 17.* Let $H : BV(\Omega) \to [\,0, +\infty\,]$ be the functional on the right-hand side of (44), i.e.,

$$H(u) = \int_\Omega \overline{f}(\nabla u)\,dx + \int_\Omega \overline{f}^\infty(\frac{D_s u}{|D_s u|})\,|D_s u|\,.$$

We have to show that $\overline{F}(u)$= (relaxed functional of $F$ with respect to the $L^1(\Omega)$-topology)= $H(u)$ for every $u \in BV(\Omega)$.

**Step 1.** $H(u) \le \overline{F}(u)$ for every $u \in BV(\Omega)$. Let us note that $\overline{f}(z) = (f(z) \wedge |z|)^{**} \le |z|$ $\forall z \in \mathbb{R}^n$. We can apply Theorem 18 with $V(z) = \overline{f}(z)$; hence

$H(u)$ is lower semicontinuous with respect to the $L^1(\Omega)$-topology on $BV(\Omega)$. Since $\overline{f}(z) \le |z|\ \forall z \in \mathbb{R}^n$ and $\overline{f}^{\infty}(z) = |z|$ we have immediately that

$$|D_s u|(\Omega) \le \begin{cases} \displaystyle\int_{S_u \cap \Omega} |u^+ - u^-|\, dH^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{on } BV(\Omega) \setminus SBV(\Omega) \end{cases}$$

and hence $H(u) \le F(u)$ for every $u \in BV(\Omega)$. By definition of relaxation we get then $H \le \overline{F}$ on $BV(\Omega)$.

**Step 2.** $H(u) \ge \overline{F}(u)$ for every $u \in BV(\Omega)$. It is enough to show that

$$\overline{F}(u) \le \begin{cases} \displaystyle\int_{\Omega} \widetilde{f}(\nabla u)\, dx & \text{if } u \in C^1(\Omega) \cap W^{1,1}(\Omega), \\ +\infty & \text{otherwise on } W^{1,1}(\Omega), \end{cases} \tag{48}$$

where

$$\widetilde{f}(z) = f(z) \wedge |z| = \begin{cases} f(z) & \text{if } z \in K, \\ |z| & \text{otherwise.} \end{cases}$$

Indeed, by Theorem 19 [28] we get then (by taking $V(z) = \widetilde{f}(z)$)

$$\overline{F}(u) \le \int_{\Omega} \overline{f}(\nabla u)\, dx \qquad \text{for every } u \in W^{1,1}(\Omega).$$

Since clearly

$$\overline{F}(u) \le \begin{cases} \displaystyle\int_{\Omega} \overline{f}(\nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise on } BV(\Omega), \end{cases}$$

by applying Theorem 18 ([39]) (with $V(z) = \overline{f}(z)$) we get finally

$$\overline{F}(u) \le H(u) \qquad \forall u \in BV(\Omega)\,.$$

To conclude the proof of Step 2 it remains therefore to prove (48). Let us consider a function $u \in C^1(\Omega) \cap W^{1,1}(\Omega)$ (hence, in particular, $u \in C^1(\Omega) \cap BV(\Omega)$). Let us consider the set

$$\Omega' = \{x \in \Omega : \nabla u(x) \in K\} = \{x \in \Omega : f(\nabla u(x)) \le |\nabla u(x)|\}\,.$$

Let us recall that if $x \in \Omega'$, then $|\nabla u(x)| \le R$ (since $K \subset B_R(0)$). For every $\varepsilon > 0$ we take an open set of finite perimeter $\Omega_\varepsilon \subset \Omega'$ such that

$$|\Omega' \setminus \Omega_\varepsilon| \le \varepsilon\,.^1$$

[1] it suffices to consider for example the set $\Omega_\varepsilon = \{x \in \Omega' : \operatorname{dist}(x, \partial\Omega') > \eta\}$ for $\eta = \eta(\varepsilon) > 0$ small enough. Remember that $\operatorname{dist}(\cdot, \partial\Omega')$ is a Lipschitz function and that $\{u > t\}$ is a set of finite perimeter for a.e. $t \in \mathbb{R}$ whenever $u$ is Lipschitz.

We construct a sequence $(u_h)$ piecewise constant on $A_\varepsilon = \Omega \setminus \overline{\Omega}_\varepsilon$ as follows: by the co-area formula (10) for any fixed $h \in \mathbb{N}$ we have

$$|Du|(A_\varepsilon) = \sum_{k\in\mathbb{Z}} \int_{\frac{k}{h}}^{\frac{k+1}{h}} H^{n-1}(\partial^*\{u>t\}\cap A_\varepsilon)\,dt\,.$$

By the mean value theorem there exists $s_k^h \in \left]\frac{k}{h}, \frac{k+1}{h}\right[$ such that

$$\frac{1}{h} H^{n-1}(\partial^*\{u>s_k^h\}\cap A_\varepsilon) \le \int_{\frac{k}{h}}^{\frac{k+1}{h}} H^{n-1}(\partial^*\{u>t\}\cap A_\varepsilon)\,dt\,. \tag{49}$$

We can then define

$$u_h = \begin{cases} \dfrac{k}{h} \text{ on } \{s_{k-1}^h < u \le s_k^h\}\cap A_\varepsilon\,,\ k\in\mathbb{Z}, \\ u \text{ on } \overline{\Omega}_\varepsilon. \end{cases} \tag{50}$$

Let us show that $u_h \in SBV(\Omega)$ :

$$\begin{aligned}
\int_{S_{u_h}\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} &= \int_{S_{u_h}\cap A_\varepsilon} \frac{1}{h}\,dH^{n-1} + \int_{(\partial^*\Omega_\varepsilon)\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} \\
&\le \sum_{k\in\mathbb{Z}} \frac{1}{h} H^{n-1}(\partial^*\{u>s_k^h\}\cap A_\varepsilon) + \frac{1}{h}\,H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega) \\
&\le \sum_{k\in\mathbb{Z}} \int_{\frac{k}{h}}^{\frac{k+1}{h}} H^{n-1}(\partial^*\{u>t\}\cap A_\varepsilon)\,dt + \frac{1}{h}\,H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega) \\
&= \int_{-\infty}^{+\infty} H^{n-1}(\partial^*\{u>t\}\cap A_\varepsilon)\,dt + \frac{1}{h}\,H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega) \\
&= \int_{A_\varepsilon} |\nabla u|\,dx + \frac{1}{h}\,H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega) \\
&\le \int_{\Omega\setminus\Omega'} |\nabla u|\,dx + \varepsilon R + \frac{1}{h}\,H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega) < +\infty\,.
\end{aligned} \tag{51}$$

Recall that $\Omega_\varepsilon$ has finite perimeter, hence $H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega)$ is finite (see Theorem 7). We can then estimate (using (51))

$$\begin{aligned}
F(u_h) &= \int_\Omega f(\nabla u_h)\,dx + \int_{S_{u_h}\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} \\
&= \int_{\Omega_\varepsilon} f(\nabla u)\,dx + \int_{S_{u_h}\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} \\
&\le \int_{\Omega'} f(\nabla u)\,dx + \int_{\Omega\setminus\Omega'} |\nabla u|\,dx + \varepsilon R + \frac{1}{h} H^{n-1}((\partial^*\Omega_\varepsilon)\cap\Omega)\,.
\end{aligned}$$

Since $u_h \to u$ in $L^\infty(\Omega)$ we obtain

$$\overline{F}(u) \le \liminf_h F(u_h) \le \int_{\Omega'} f(\nabla u)\,dx + \int_{\Omega\setminus\Omega'} |\nabla u|\,dx + \varepsilon R$$

for any $\varepsilon > 0$. By the arbitrariness of $\varepsilon$ we conclude that for every $u \in C^1(\Omega) \cap W^{1,1}(\Omega)$ we have

$$\overline{F}(u) \leq \int_\Omega \widetilde{f}(\nabla u)\, dx\,, \qquad \text{with } \widetilde{f}(z) = f(z) \wedge |z|,$$

and (48) follows. □

## 7 Lower semicontinuity and relaxation in higher dimension and for vector-valued functions

Let us start with some notation and definitions not given previously. We shall denote by $\mathbb{M}^{k\times n}$ the space of $k \times n$ matrices ($k$ rows, $n$ columns) and by $\mathbb{M}_1^{k\times n}$ the subset of $\mathbb{M}^{k\times n}$ of all matrices with rank less than or equal to one. We shall identify $\mathbb{M}^{k\times n}$ with $\mathbb{R}^{kn}$. If $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^n$, the tensor product $a \otimes b \in \mathbb{M}_1^{k\times n}$ is the matrix whose entries are $a_i b_j$ with $i = 1, \ldots, k$, and $j = 1, \ldots, n$. Conversely, if a matrix $\xi$ has rank one, there are two vectors $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^n$ such that $\xi = a \otimes b$. If $A \subset \mathbb{R}^k$ and $b \in \mathbb{R}^n$, we set $A \otimes b = \{a \otimes b : a \in A\} \subset \mathbb{M}_1^{k\times n}$. Remark that $|a \otimes b| = |a||b|$ (the norms are taken in the proper spaces).

**Definition 12.** (see [40, 41, 31, 32, 23]) *We say that a continuous function* $\varphi : \mathbb{M}^{k\times n} \to [\,0, +\infty[$ *is* quasiconvex *if*

$$|A|\varphi(\xi) \leq \int_A \varphi(\xi + \nabla u(x))\, dx \tag{52}$$

*for every* $\xi \in \mathbb{M}^{k\times n}$, *A bounded open subset of* $\mathbb{R}^n$ *and* $u \in C_c^1(A; \mathbb{R}^k)$.

This property is a well-known necessary and sufficient condition for the lower semicontinuity of multiple integrals in Sobolev spaces (see [1], [31]) playing the same role of the convexity in the case of scalar functions (condition (52) is equivalent to convexity in the case $k = 1$). Let us recall that if we allow the function $u$ to be vector-valued, i.e., $u \in W^{1,p}(\Omega; \mathbb{R}^k)$, then the convexity hypothesis turns out to be sufficient, but too strong to be necessary for $F(u) = \displaystyle\int_\Omega f(\nabla u)\, dx$ to be weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^k)$.

*Remark 23.* Every quasiconvex function $\varphi : \mathbb{M}^{k\times n} \to [\,0, +\infty[$ is *rank-one convex*, i.e., $\varphi$ satisfies

$$\varphi(\lambda\xi + (1-\lambda)\eta) \leq \lambda\varphi(\xi) + (1-\lambda)\varphi(\eta)$$

for every $\xi$, $\eta \in \mathbb{M}^{k\times n}$ such that $\operatorname{rank}(\xi - \eta) \leq 1$, and every $\lambda \in [0,1]$ (see [31], [32], [23]). V. Šverák shows in [45] that the converse is not true.

Before we start with a lower semicontinuity result on $SBV(\Omega;\mathbb{R}^k)$ for functionals of the form

$$F(u) = \int_\Omega f(\nabla u)\,dx + \int_{S_u\cap\Omega} g(u^+ - u^-, \nu_u)\,dH^{n-1}\,, \tag{53}$$

let us introduce here briefly the space $BV(\Omega;\mathbb{R}^k)$ and $SBV(\Omega;\mathbb{R}^k)$. All the notions and properties (like $S_u$, approximate limit, approximate differential, rectifiability of $S_u, \ldots$) which can be extended naturally from $k = 1$ to $k > 1$ will be omitted (see, for example, [13]).

We say that $u \in L^1(\Omega;\mathbb{R}^k)$ belongs to $BV(\Omega;\mathbb{R}^k)$ if for any $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n\}$ there is a measure $\mu_i^j$ with finite total variation in $\Omega$ such that

$$\int_\Omega u^{(i)} \frac{\partial\phi}{\partial x_j}\,dx = -\int_\Omega \phi\,d\mu_i^j \qquad \forall\phi \in C_c^1(\Omega)\,.$$

We denote by $Du$ the $\mathbb{M}^{k\times n}$-valued measure whose components are the $\mu_i^j$, and by $|Du|$ its total variation. Also for a function $u \in BV(\Omega;\mathbb{R}^k)$ we have the Lebesgue decomposition

$$Du = D_a u + D_s u = \nabla u L^n + D_s u\,,$$

where we denote by $\nabla u$ the density of the absolutely continuous part of $Du$ with respect to the Lebesgue measure. The singular part of $Du$ with respect to the Lebesgue measure can be further decomposed into two mutually singular measures as

$$D_s u = (u^+ - u^-) \otimes \nu_u H^{n-1} \llcorner S_u + D_c u\,,$$

where $(u^+ - u^-) \otimes \nu_u H^{n-1} \llcorner S_u$ is the Hausdorff part and $D_c u$ the Cantor part of Du. We indicate by $\dfrac{D_s u}{|D_s u|}$ the Radon-Nikodým derivative of $D_s u$ with respect to its total variation. We say that $u \in SBV(\Omega;\mathbb{R}^k)$ if $u \in BV(\Omega;\mathbb{R}^k)$ and $D_c u = 0$.

**Theorem 20 (Lower semicontinuity for functionals on $SBV(\Omega;\mathbb{R}^k)$).** *Let $f : \mathbb{M}^{k\times n} \to [\,0,+\infty[$ be quasiconvex, and satisfy a growth condition of order $p > 1$, i.e.,*

$$|z|^p \le f(z) \le c(1 + |z|^p) \qquad \forall z \in \mathbb{M}^{k\times n}\,. \tag{54}$$

*Let $g : \mathbb{M}^{k\times n} \to [\,0,+\infty[$ be of the form*

$$g(s,\nu) = \theta(|s|)\psi(\nu) \qquad \forall s \in \mathbb{R}^k\,,\ \forall \nu \in \mathbb{R}^n \tag{55}$$

*satisfying*

$$g(s,\nu) \ge c > 0\,, \tag{56}$$

*with $\theta : \,]0,+\infty[\,\to\,]0,+\infty[$ concave, non decreasing, and $\psi : \mathbb{R}^n \to [\,0,+\infty[$ convex, even, positively 1-homogeneous. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Then the functional defined in (53), i.e.,*

$$F(u) = \int_{\Omega} f(\nabla u)\, dx + \int_{S_u \cap \Omega} g(u^+ - u^-, \nu_u)\, dH^{n-1}$$

*is* $L^1_{\text{loc}}(\Omega;\mathbb{R}^k)$*-lower semicontinuous on* $SBV(\Omega;\mathbb{R}^k)$*.*

The proof of this result can be found in [7].

*Remark 24 (Relaxation on* $SBV(\Omega;\mathbb{R}^k)$ *- the superlinear growth case).* Notice that a corresponding relaxation result has been obtained by Fonseca and Francfort in [37]: they study the relaxation with respect to the $L^1(\Omega;\mathbb{R}^k)$-topology in $SBV(\Omega;\mathbb{R}^k)$ of the functional

$$\int_{\Omega} f(\nabla u)\, dx + \lambda H^{n-1}(S_u \cap \Omega) \qquad \lambda > 0 \tag{57}$$

(note: $g(s,\nu) \equiv \lambda$), where $f$ has $p$-growth ($p > 1$) and satisfies a local Lipschitz condition. The relaxed functional related to (57) is defined only on $SBV(\Omega;\mathbb{R}^k)$ and has the form

$$\int_{\Omega} Qf(\nabla u)\, dx + \lambda H^{n-1}(S_u \cap \Omega)\,,$$

where $Qf$ is the quasiconvexification of $f$ given by

$$Qf(\xi) = \inf\{\int_{\Omega} f(\xi + \nabla u(x))\, dx : u \in C^1_c(\Omega;\mathbb{R}^k)\}\,.$$

Let us note that if $f : \mathbb{M}^{k\times n} \to \mathbb{R}$ is a locally bounded Borel function, then

$$Qf = \sup\{\varphi : \mathbb{M}^{k\times n} \to \mathbb{R} : \varphi \text{ quasiconvex}, \varphi \leq f\}$$

is the quasiconvex envelope of $f$.

Without assuming any local Lipschitz condition on $f$ in (57) a completely analogous result has been proved in [20].

*Remark 25 (Relaxation on* $SBV(\Omega;\mathbb{R}^k)$ *- the linear growth case: interaction between bulk and surface energy densities).* In [14], Barroso, Bouchittè, Buttazzo and Fonseca proved a relaxation result for (53) (the integrand functions $f$ and $g$ possibly depending also on the $x$-variable) with respect to the $L^1$-convergence in the space $SBV(\Omega;\mathbb{R}^k)$, where they assumed that $f$ is quasiconvex, and has linear growth, and that $g(\cdot,\nu)$ grows also linearly, i.e., $c|\xi| \leq f(\xi) \leq C(1+|\xi|)$, and $c_1|\eta| \leq g(\eta,\nu) \leq C_1|\eta|$ (when $g \equiv 0$, i.e., the relaxation of quasiconvex functionals in $BV(\Omega;\mathbb{R}^k)$ for integrands $f(x,u,\nabla u)$ has been performed by Fonseca and Müller (see [38])). This result extends "partially" the paper by [17] since they are working with linear growth assumptions on $f$ or $g$, but they had to make an isotropy assumption on $f$ and $g$ (the result of this paper will be stated in Section 8).

In the remaining part of this section we will present a relaxation result in $BV(\Omega;\mathbb{R}^k)$ with respect to the $L^1$-topology of the functional

$$F(u) = \int_{\Omega} f(\nabla u)\,dx + \int_{S_u\cap\Omega} g((u^+ - u^-)\otimes \nu_u)\,dH^{n-1}\,,$$

obtained by Braides and Coscia in [22] under very mild assumptions on $f$ and under the assumption that $g$ is positively 1-homogeneous.

The main motivation to present the proof of this relaxation result lies in the desire to show an often used scheme of proof followed in this context (see Remark 13) forced by the fact that for vector-valued functions no co-area formula is available.

We will state the theorem with a stricter assumption on $g$ as that given in the original paper. In other words, instead of requiring $g$ locally bounded on rank-one matrices (i.e., on $\mathbb{M}_1^{k\times n}$), one gets the same relaxation result under the hypothesis that there exist $n$ linearly independent vectors $\nu_1,\dots,\nu_n$ in $S^{n-1}$ such that $g$ is locally bounded on $\mathbb{R}^k\otimes\nu_m$ for all $m=1,\dots,n$.

**Theorem 21 (Relaxation in $BV(\Omega;\mathbb{R}^k)$ - no growth assumptions on the bulk energy and the surface energy densities: interaction of the two energy densities by relaxation).** *Let $f:\mathbb{M}^{k\times n}\to[0,+\infty]$ be a positive Borel function such that $f(0)\neq+\infty$; let $g:\mathbb{M}_1^{k\times n}\to[0,+\infty[$ be a positively 1-homogeneous Borel function, locally bounded on $\mathbb{M}_1^{k\times n}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $F: BV(\Omega;\mathbb{R}^k)\to[0,+\infty]$ be the functional defined by*

$$F(u) = \begin{cases} \displaystyle\int_{\Omega} f(\nabla u)\,dx + \int_{S_u\cap\Omega} g((u^+ - u^-)\otimes\nu_u)\,dH^{n-1} & \text{if } u\in SBV(\Omega;\mathbb{R}^k)\\ +\infty & \text{otherwise.}\end{cases} \tag{58}$$

*Then the relaxed functional with respect to the $L^1(\Omega;\mathbb{R}^k)$-topology is given by*

$$\overline{F}(u) = \int_{\Omega}\varphi(\nabla u)\,dx + \int_{\Omega}\varphi^{\infty}\Big(\frac{D_s u}{|D_s u|}\Big)\,|D_s u| \tag{59}$$

*for every $u\in BV(\Omega;\mathbb{R}^k)$, where the function $\varphi:\mathbb{M}^{k\times n}\to[0,+\infty[$ is given by*

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex}, \psi\le f \text{ on } \mathbb{M}^{k\times n}, \psi^{\infty}\le g \text{ on } \mathbb{M}_1^{k\times n}\} \tag{60}$$

*and satisfies $0\le\varphi(\xi)\le c(1+|\xi|)$ for every $\xi\in\mathbb{M}^{k\times n}$.*

*Remark 26.* If we suppose in addition that $f$ is convex, and $g$ is rank-one convex on $\mathbb{R}^k\otimes\nu$ for every $\nu\in S^{n-1}$ (i.e., $g(\lambda a\otimes\nu + (1-\lambda)b\otimes\nu)\le \lambda g(a\otimes\nu) + (1-\lambda)g(b\otimes\nu), \forall a,b\in\mathbb{R}^k, \lambda\in[0,1], \nu\in S^{n-1}$), then $F$ is convex on $SBV(\Omega;\mathbb{R}^k)$, and we get that also the relaxed functional $\overline{F}$ must be convex, and hence also the integrand $\varphi$. Then we have the formula

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ convex}, \psi\le f \text{ on } \mathbb{M}^{k\times n}, \psi^{\infty}\le g \text{ on } \mathbb{M}_1^{k\times n}\}\,.$$

In the scalar case ($k = 1$) we get as a corollary of Theorem 21 a generalization of the relaxation theorem 17, where $\varphi(z) = (f(z) \wedge |z|)^{**}$ (in this case we supposed $f(0) = 0$.)

**Corollary 2.** *Let $k = 1$. Under the hypotheses of Theorem 21, $\varphi$ satisfies the formula*

$$\varphi(z) = (f \wedge (f(0) + g))^{**}(z) \qquad \forall z \in \mathbb{R}^n .$$

*Proof.* Let $\psi$ be a convex function such that $\psi \le f$ and $\psi^\infty \le g$ on $\mathbb{R}^n$. In particular $\psi(0) \le f(0)$, and we get then that

$$\psi \le f(0) + g . \tag{61}$$

Indeed, for $0 < t \le 1$, $z \in \mathbb{R}^n$, using the convexity of $\psi$

$$\psi(z) = \psi(t\frac{z}{t}) \le t\psi(\frac{z}{t}) + \psi(0) \le t\psi(\frac{z}{t}) + f(0) .$$

By passing to the limit as $t \to 0^+$, we get $\psi(z) \le \psi^\infty(z) + f(0) \le g(z) + f(0)$ proving (61). By Remark 26 we conclude that

$$\varphi(z) = \sup\{\psi(z) : \psi \text{ convex}, \psi \le f \wedge (f(0) + g)\} = (f \wedge (f(0) + g))^{**}(z)$$

as desired. $\square$

*Remark 27.* Let $H : BV(\Omega; \mathbb{R}^k) \to [0, +\infty[$ be the functional on the right-hand side of (59), i.e.,

$$H(u) = \int_\Omega \varphi(\nabla u)\, dx + \int_{S_u \cap \Omega} \varphi^\infty(\frac{D_s u}{|D_s u|})\, |D_s u|$$

with $\varphi$ given by (60). We have to show that $\overline{F}(u)$=(relaxed functional of $F$ with respect to the $L^1(\Omega; \mathbb{R}^k)$-topology)= $H(u)$ for every $u \in BV(\Omega; \mathbb{R}^k)$.

Analogously to the other proofs of relaxation we will show that $H \le \overline{F}$ and $H \ge \overline{F}$. While the proof of the lower bound (i.e., $H \le \overline{F}$) is completely analogous to the previous ones (based on a semicontinuity result for $H$ and the inequality $H \le F$ which can be proved using the definition of $\varphi$ in (60)), the proof of the upper bound (i.e., $\overline{F} \le H$) will be completely different and we have to work hard:

**7.1:** we will first establish that under the very weak hypotheses on $f$ and $g$ the relaxed functional $\overline{F}$ has linear growth, and hence, is finite on the whole $BV(\Omega; \mathbb{R}^k)$ ;

**7.2:** then we will prove by a measure theoretical approach and a localization technique that the study of the relaxed functional at a fixed $u$ can be reduced to the study of a regular Borel measure on $\Omega$ ;

**7.3:** this fact allows us to use some integral representation arguments of Buttazzo and Dal Maso and to write the restriction of $\overline{F}$ to $W^{1,1}(\Omega;\mathbb{R}^k)$ as an integral;

**7.4:** the use of a relaxation result of Ambrosio and Dal Maso together with the formula (60) and some suitable estimates on the above integrand functions will finally give $\overline{F} \leq H$ on $BV(\Omega;\mathbb{R}^k)$.

Let us state in the sequel the integral representation theorem for functionals defined on Sobolev spaces we will use. Moreover let us recall a lower semicontinuity and relaxation theorem for quasiconvex integrals on the space of vector-valued $BV$-functions (this last result is a generalization to quasiconvex functions of the result by Goffman and Serrin used in Section 5 and Section 6).

**Theorem 22 (Integral representation theorem on Sobolev spaces).** (see [28]) *Let $F : W^{1,1}(\Omega;\mathbb{R}^k) \times A(\Omega) \to [\,0,+\infty[$ be a functional satisfying for every $u\,, v \in W^{1,1}(\Omega;\mathbb{R}^k)$ and for every $A \in A(\Omega)$ the following properties:*

*(i) (linear growth)* $\quad |F(u,A)| \leq c(|A| + \int_A |\nabla u(x)|\,dx)\,;$

*(ii) (locality)* $\quad F(u,A) = F(v,A)$ *whenever* $u = v$ *on* $A\,;$

*(iii) (semicontinuity)* $\quad F(\cdot,A)$ *is* $W^{1,1}$*-sequentially lower semicontinuous;*

*(iv) (translation invariance)* $\quad F(u+b) = F(u,A)$ *for every constant vector* $b \in \mathbb{R}^k\,;$

*(v)* $F(u,\cdot)$ *is the restriction to* $A(\Omega)$ *of a regular Borel measure.*

*Then there exists a Carathéodory function $\psi : \Omega \times \mathbb{M}^{k\times n} \to [\,0,+\infty[\,$, quasiconvex in the second variable for a.e. $x \in \Omega$, such that the integral representation*

$$F(u,A) = \int_A \psi(x,\nabla u(x))\,dx$$

*holds for every $u \in W^{1,1}(\Omega;\mathbb{R}^k)$ and for every $A \in A(\Omega)$.*

**Theorem 23 (Semicontinuity on $BV(\Omega;\mathbb{R}^k)$ - Relaxation).** (see [12]) *Let $\varphi : \mathbb{M}^{k\times n} \to [\,0,+\infty[$ be a quasiconvex function satisfying*

$$0 \leq \varphi(\xi) \leq c(1+|\xi|) \qquad \forall \xi \in \mathbb{M}^{k\times n}\,. \tag{62}$$

*Let $\mathcal{F} : BV(\Omega;\mathbb{R}^k) \to [\,0,+\infty\,]$ be the functional defined by*

$$\mathcal{F}(u) = \int_\Omega \varphi(\nabla u(x))\,dx + \int_\Omega \varphi^\infty\big(\frac{D_s u}{|D_s u|}\big)\,|D_s u|\,. \tag{63}$$

*Then*

*(i) $\mathcal{F}$ is $L^1(\Omega;\mathbb{R}^k)$-lower semicontinuous on $BV(\Omega;\mathbb{R}^k)\,$;*

*(ii) $\mathcal{F} = \overline{\mathcal{F} + \chi_{W^{1,1}(\Omega;\mathbb{R}^k)}}$, i.e., $\mathcal{F}$ coincides with the relaxation of its restriction to the Sobolev space $W^{1,1}(\Omega;\mathbb{R}^k)$.*

*Remark 28.* Note that in order to have a good definition of $\overline{F}$ in (59) and of $\mathcal{F}$ in (63) we have to extend the notion of $\varphi^\infty$ to $\varphi$ quasiconvex. This quantity is well-defined by $\lim_{t\to+\infty}\dfrac{\varphi(t\xi)}{t}$ if $\mathrm{rank}(\xi)\le 1$, since quasiconvex functions are convex in rank-one directions. In general, the above limit does not exist for all $\xi\in\mathbb{M}^{k\times n}$ (see [42]). A more recent result by Alberti (see [2]) assures that the matrix $\dfrac{D_s u}{|D_s u|}$ is of rank 1 $|D_s u|$-a.e. , and hence $\overline{F}$ and $\mathcal{F}$ in (59) and in (63) respectively, are well defined.

### 7.1 Upper bound for the relaxed functional

We start by giving an upper bound for the relaxed functional $\overline{F}$. We shall consider the functional $G : BV(\Omega;\mathbb{R}^k)\to[\,0,+\infty\,]$ defined as

$$G(u)=\begin{cases}\displaystyle\int_\Omega f(\nabla u(x))\,dx+\int_{S_u\cap\Omega} g((u^+-u^-)\otimes\nu_u)\,dH^{n-1} \\ \qquad\qquad\qquad\text{if } u\in SBV(\Omega;\mathbb{R}^k) \text{ and } H^{n-1}(S_u\cap\Omega)<+\infty,\\ +\infty \qquad\qquad\qquad \text{otherwise on } BV(\Omega;\mathbb{R}^k).\end{cases} \tag{64}$$

Clearly, $F\le G$. Obviously an upper bound of $\overline{G}=$ (relaxed functional of $G$ with respect to the $L^1$-topology in $BV$) will do as well.

**Proposition 4 (linear growth).** *We have*

$$\overline{G}(u)\le c(1+|Du|(\Omega))\qquad \forall u\in BV(\Omega;\mathbb{R}^k)\,.$$

*Proof.* Since $g$ is locally bounded on $\mathbb{M}_1^{k\times n}$ we set

$$M=\sup\{g(a\otimes\nu): a\in S^{k-1}\,,\nu\in S^{n-1}\}<+\infty\,. \tag{65}$$

**Step 1:** $k=1$**.** In this case $g$ is defined on the whole $\mathbb{R}^n=\mathbb{R}\otimes\mathbb{R}^n$. Let us consider $u\in BV(\Omega)\cap C^1(\Omega)$. Let us fix $h\in\mathbb{N}$. By the co-area formula (10) we have

$$|Du|(\Omega)=\sum_{j\in\mathbb{Z}}\int_{\frac{j}{h}}^{\frac{j+1}{h}} H^{n-1}(\partial^*\{u>t\}\cap\Omega)\,dt\,.$$

Hence, by the mean value theorem, for every $j\in\mathbb{Z}$ we can find

$$s_j^h\in\,\Big]\frac{j}{h},\frac{j+1}{h}\Big[$$

such that

$$\frac{1}{h}H^{n-1}(\partial^*\{u>s_j^h\}\cap\Omega)\le\int_{\frac{j}{h}}^{\frac{j+1}{h}} H^{n-1}(\partial^*\{u>t\}\cap\Omega)\,dt\,,$$

so that

$$\sum_{j\in\mathbb{Z}} \frac{1}{h} H^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \le |Du|(\Omega)\,. \tag{66}$$

Let us now construct the sequence $(u_h)$ in $SBV(\Omega)$ as follows:

$$u_h(x) = \frac{j}{h} \qquad \text{on } \ \{s_{j-1}^h < u < s_j^h\}. \tag{67}$$

It is clear that for every $h \in \mathbb{N}$ we have $\nabla u_h(x) = 0$ for a.e. $x \in \Omega$,

$$S_{u_h} \cap \Omega = \bigcup_{j\in\mathbb{Z}} \big(\partial^*\{u > s_j^h\} \cap \Omega\big)\,,$$

and

$$Du_h = D_s u_h = \sum_{j\in\mathbb{Z}} \frac{1}{h} \nu_h^j H^{n-1} \llcorner \partial^*\{u > s_j^h\} \cap \Omega\,,$$

where $\nu_h^j$ is defined by

$$D\mathbf{1}_{\{u > s_j^h\}} = \nu_h^j H^{n-1} \llcorner \partial^*\{u > s_j^h\}\,.$$

Hence we obtain

$$\begin{aligned} F(u_h) &= \int_\Omega f(\nabla u_h(x))\,dx + \int_{S_{u_h}\cap\Omega} g((u_h^+ - u_h^-)\otimes \nu_{u_h})\,dH^{n-1} \\ &= f(0)|\Omega| + \sum_{j\in\mathbb{Z}} \int_{\partial^*\{u>s_j^h\}\cap\Omega} g(\frac{1}{h}\nu_h^j)\,dH^{n-1} \\ &\le f(0)|\Omega| + \sum_{j\in\mathbb{Z}} \frac{1}{h} M H^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \\ &\le f(0)|\Omega| + M|Du|(\Omega)\,. \end{aligned}$$

We have made use of the positive homogeneity of $g$, of (65) and (66). Remark also that by (66) for every $h \in \mathbb{N}$ we get

$$H^{n-1}(S_{u_h} \cap \Omega) = \sum_{j\in\mathbb{Z}} H^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \le h|Du|(\Omega) < +\infty;$$

hence

$$|Du_h|(\Omega) = \int_{S_{u_h}\cap\Omega} |u_h^+ - u_h^-|\,dH^{n-1} = \frac{1}{h} H^{n-1}(S_{u_h} \cap \Omega) \le |Du|(\Omega)$$

and $G(u_h) = F(u_h)$ (note that we have proved $H^{n-1}(S_{u_h} \cap \Omega) < +\infty$). Since $(u_h)$ converges to $u$ in $L^\infty(\Omega)$, by the definition of $\overline{G}$ we conclude that

$$\overline{G}(u) = \liminf_{h\to+\infty} G(u_h) \le f(0)|\Omega| + M|Du|(\Omega)\,.$$

For a general $u \in BV(\Omega)$ it suffices to recall that there exists a sequence $(v_h)$ in $C^\infty(\Omega) \cap BV(\Omega)$ such that $v_h \to u$ in $L^1(\Omega)$ and

$$|Du|(\Omega) = \lim_{h\to+\infty} |Dv_h|(\Omega) = \lim_{h\to+\infty} \int_\Omega |\nabla v_h|\, dx\,.$$

By the lower semicontinuity of $\overline{G}$ we get

$$\overline{G}(u) \le \liminf_{h\to+\infty} \overline{G}(v_h) \le \lim_{h\to+\infty} \big(f(0)|\Omega| + M|Dv_h|(\Omega)\big) = f(0)|\Omega| + M|Du|(\Omega)\,.$$

**Step 2:** $k \ge 2$**.** We can proceed "componentwise": given a function $u = (u_{(1)}, \dots, u_{(k)}) \in C^1(\Omega;\mathbb{R}^k) \cap BV(\Omega;\mathbb{R}^k)$ we apply the same procedure as in Step 1 to every $h \in \mathbb{N}$, $j \in \mathbb{Z}$ and $i = 1, \dots, k$ (for more details see [22], Proposition 3.1). We obtain then by approximation

$$\overline{G}(u) \le f(0)|\Omega| + \sqrt{k} M|Du|(\Omega)$$

for every $u \in BV(\Omega;\mathbb{R}^k)$. □

*Remark 29.* Let us point out that in Step 1 of the previous proof we have shown that fixed $u \in BV(\Omega) \cap C^1(\Omega)$ there exists a sequence $(u_h) \subset SBV(\Omega)$ such that $u_h \to u$ in $L^\infty(\Omega)$, $\nabla u_h = 0$ a.e. on $\Omega$, $H^{n-1}(S_{u_h} \cap \Omega) < +\infty$, and $|Du_h|(\Omega) \le |Du|(\Omega)$. By using the density of $C^1(\Omega) \cap BV(\Omega)$ in $BV(\Omega)$ and a diagonalization argument we can then assert that for every $u \in BV(\Omega)$ there exists a sequence $(u_h) \subset SBV(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$, with $\nabla u_h = 0$ a.e. on $\Omega$, $H^{n-1}(S_{u_h} \cap \Omega) < +\infty$, and $\lim_{h\to+\infty} |Du_h|(\Omega) = |Du|(\Omega)$.

## 7.2 De Giorgi-Letta criterion to prove measure properties of the relaxed functional

Our goal is to apply to the "localized version" of the functional $\overline{G}$ the integral representation theorem 22. The main difficulty is now to prove hypothesis (v) of Theorem 22 for $\overline{G}(u, \cdot)$. To this end we will use the following theorem by De Giorgi and Letta, which provides a sufficient condition to obtain a (regular) Borel measure starting with a (regular) increasing set function defined only on $A(\Omega)$.

**Theorem 24 (De Giorgi and Letta criterion).** (see [35]) *Let $\mu : A(\Omega) \to [\,0, +\infty\,]$ be a positive set function defined on $A(\Omega)$. Assume*

*(i)* $\mu(\emptyset) = 0$, $\mu(A) \le \mu(B)$ *if* $A \subseteq B$ *($\mu$ is an increasing set function);*
*(ii)* $\mu(A \cup B) \le \mu(A) + \mu(B)$ $\forall A, B \in A(\Omega)$ *($\mu$ is subadditive);*
*(iii)* $\mu(A \cap B) \ge \mu(A) + \mu(B)$ $\forall A, B \in A(\Omega)$ *with* $A \cap B = \emptyset$ *($\mu$ is superadditive on disjoint sets);*
*(iv)* $\mu(A) = \sup\{\mu(B) : B \subset\subset A, B \in A(\Omega)\}$ *($\mu$ is regular).*

*Then the extension $\mu$ to $B(\Omega)$ given by*

$$\mu(B) = \inf\{\mu(A) : A \in A(\Omega)\,, B \subseteq A\} \tag{68}$$

*is a positive (regular) Borel measure.*

For a proof of the above theorem we refer also to [33], Theorem 14.23, [13], Theorem 1.53, [23], Theorem 10.2.

Let us finally recall the *Fleming-Rishel co-area formula.* Let $u$ be a Lipschitz function; then for every $v \in BV(\Omega)$ we have

$$\int_\Omega v|\nabla u|\,dx = \int_{-\infty}^{+\infty} \int_{\partial^*\{u>t\}\cap\Omega} \widetilde{v}\,dH^{n-1}\,dt \tag{69}$$

($\nabla u$ is the a.e. gradient of $u$, and $\widetilde{v}$ denotes the approximate limit of $v$).

Let us localize the functional $G$ introduced by (64). We consider $G(u, A) : BV(\Omega; \mathbb{R}^k) \times A(\Omega) \to [\,0, +\infty\,]$ given by

$$G(u,A) = \begin{cases} \displaystyle\int_A f(\nabla u(x))\,dx + \int_{S_u\cap A} g((u^+ - u^-)\otimes \nu_u)\,dH^{n-1} \\ \qquad\qquad\qquad \text{if } u \in SBV(\Omega;\mathbb{R}^k) \text{ and } H^{n-1}(S_u\cap\Omega) < +\infty, \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise on } BV(\Omega;\mathbb{R}^k). \end{cases} \tag{70}$$

Obviously, $G(u, \Omega) = G(u)$. We define also

$$\overline{G}(u, A) = \inf\{\liminf_{h\to+\infty} G(u_h, A) : u_h \to u \text{ in } L^1(A; \mathbb{R}^k), u_h \in BV(\Omega; \mathbb{R}^k)\}\,.$$

Localizing the proof of Proposition 4 we get the linear growth condition

$$\overline{G}(u, A) \leq c(|A| + |Du|(A)) \tag{71}$$

for every $u \in BV(\Omega; \mathbb{R}^k)$, and for every $A \in A(\Omega)$.

**Proposition 5.** *For every $u \in BV(\Omega; \mathbb{R}^k)$ the set function $\overline{G}(u, \cdot)$ is (the restriction to $A(\Omega)$ of) a regular Borel measure on $\Omega$.*

*Proof.*

**Step 1.** It is immediate to show that $\overline{G}(u, \cdot)$ is a positive, increasing set function.

**Step 2.** $\overline{G}(u, \cdot)$ is regular; i.e., for every open set $A \subset \Omega$, we have

$$\overline{G}(u, A) = \sup\{\overline{G}(u, A') : A' \text{ open}\,, A' \subset\subset A\}\,. \tag{72}$$

Indeed, since $\overline{G}(u, \cdot)$ is an increasing function, the inequality " $\geq$ " in (72) is trivial. Let us now prove the opposite inequality. Fixed a compact set $K$ in $A$, let us define

$$\delta = \frac{1}{2}\mathrm{dist}(\partial A, K) \qquad\qquad d_K(x) = \mathrm{dist}(x, K)$$

and

$$B_t = \{x \in A : d_K(x) < t\} \ \text{ with } t \in \,]0, \delta[$$

$$B = B_\delta = \{x \in A : d_K(x) < \delta\}\,.$$

Note that for a.e. $t \in \,]0, \delta[\,$, $B_t$ is a set of finite perimeter. By definition of relaxation, there exist two sequences $(u_h)$, $(v_h) \in SBV(\Omega; \mathbb{R}^k)$ such that $H^{n-1}(S_{u_h} \cap \Omega) < +\infty, H^{n-1}(S_{v_h} \cap \Omega) < +\infty$, and

$$\begin{array}{ll} u_h \to u & \text{in } L^1(B; \mathbb{R}^k) \\ v_h \to u & \text{in } L^1(A \setminus K; \mathbb{R}^k) \end{array}$$

and

$$\overline{G}(u, B) = \lim_{h \to +\infty} G(u_h, B) \qquad \overline{G}(u, A \setminus K) = \lim_{h \to +\infty} G(v_h, A \setminus K)\,.$$

Since we have $H^{n-1}(S_{u_h} \cap \Omega) + H^{n-1}(S_{v_h} \cap \Omega) < +\infty$, we get that

$$H^{n-1}(S_{u_h} \cap \partial^* B_t) + H^{n-1}(S_{v_h} \cap \partial^* B_t) = 0 \tag{73}$$

for a.e. $t \in \,]0, \delta[\,$ (for a proof see Appendix B). We can apply the Fleming-Rishel co-area formula (69) with $v = |u_h - v_h|$ and $u(x) = d_K(x)$ (recall that $|\nabla d_K(x)| = 1$ a.e.); we get

$$\int_{B \setminus K} |u_h - v_h|\, dx = \int_0^\delta \int_{\partial^* B_t} |\widetilde{u}_h(x) - \widetilde{v}_h(x)|\, dH^{n-1}(x)\, dt\,.^2$$

By the mean value theorem (see [3] below), for every $h \in \mathbb{N}$ we can choose $t_h \in \,]0, \delta[\,$ such that (73) holds, $B_{t_h}$ is a set of finite perimeter, and

$$\int_{\partial^* B_t} |\widetilde{u}_h - \widetilde{v}_h|\, dH^{n-1} \leq \frac{1}{\delta} \int_{B \setminus K} |u_h - v_h|\, dx\,. \tag{74}$$

Let us now construct a "suitable" sequence $w_h$ in order to estimate $\overline{G}(u, A)$. We can define the sequence $(w_h)$ in $L^1(\Omega; \mathbb{R}^k)$ by setting

$$w_h = \begin{cases} u_h \text{ in } B_{t_h} \\ \\ v_h \text{ in } \ \Omega \setminus B_{t_h} \end{cases}$$

(It is convenient to make a "cut" between minimizing sequences! Since $B_{t_h}$ is a set of finite perimeter we have in particular $H^{n-1}(\partial^* B_{t_h}) = 0$, hence

[2] The functions $\widetilde{u}_h$ and $\widetilde{v}_h$ are the approximate limits of $u_h$ and $v_h$, respectively. They exist $H^{n-1}$-a.e. on $\partial^* B_t$ since (73) holds.

[3] Let $f \in L^1(a, b)$; then $L^n(\{t \in \,]a, b[\, : f(t) \leq \frac{1}{b-a} \int_a^b f(s)\, ds\}) > 0$.

$L^n(\partial^* B_{t_h}) = 0$. Therefore $w_h$ is defined a.e. on $\Omega$). Note that for every $h$ we have

$$w_h \in SBV(\Omega;\mathbb{R}^k)\,, \qquad\qquad H^{n-1}(S_{w_h} \cap \Omega) < +\infty\,.$$

Moreover on $A$ we have

$$\nabla w_h = \nabla u_h \mathbf{1}_{B_{t_h}} + \nabla v_h \mathbf{1}_{A\setminus B_{t_h}}\,,$$

and the Hausdorff part of the measure $Dw_h$ is given by

$$\begin{aligned}&(u_h^+ - u_h^-) \otimes \nu_{u_h} H^{n-1} \llcorner S_{u_h} \cap B_{t_h} + \\ &+ (v_h^+ - v_h^-) \otimes \nu_{v_h} H^{n-1} \llcorner S_{v_h} \cap (A \setminus B_{t_h}) + \\ &+ (\widetilde{u}_h - \widetilde{v}_h) \otimes \nu_{\partial^* B_{t_h}} H^{n-1} \llcorner \partial^* B_{t_h}\,,\end{aligned}$$

where $\nu_{\partial^* B_{t_h}}$ denotes the normal to $\partial^* B_{t_h}$ pointing inwards $B_{t_h}$ (by (73) we do not have jumps for $w_h$ on $\partial^* B_{t_h}$). We then obtain (using (65), the positive 1-homogeneity of $g$ and (74))

$$\begin{aligned}G(w_h, A) &= G(w_h, B_{t_h}) + G(w_h, A \setminus \overline{B}_{t_h}) \\ &\le G(u_h, B) + G(v_h, A \setminus K) + \int_{\partial^* B_{t_h}} g((w_h^+ - w_h^-) \otimes \nu_{\partial^* B_{t_h}})\, dH^{n-1} \\ &\le G(u_h, B) + G(v_h, A \setminus K) + M \int_{\partial^* B_{t_h}} |w_h^+ - w_h^-|\, dH^{n-1} \\ &\le G(u_h, B) + G(v_h, A \setminus K) + M \int_{\partial^* B_{t_h}} |\widetilde{u}_h - \widetilde{v}_h|\, dH^{n-1} \\ &\le G(u_h, B) + G(v_h, A \setminus K) + \tfrac{M}{\delta} \int_{B\setminus K} |u_h - v_h|\, dH^{n-1}\,.\end{aligned}$$

Since $w_h \to u$ in $L^1(A;\mathbb{R}^k)$ and $(u_h - v_h) \to 0$ in $L^1(B \setminus K;\mathbb{R}^k)$, we have, by taking the limit as $h \to +\infty$,

$$\overline{G}(u, A) \le \liminf_{h\to+\infty} G(w_h, A) \le \overline{G}(u, B) + \overline{G}(u, A \setminus K)\,.$$

By (71) we have then

$$\overline{G}(u, A) \le \overline{G}(u, B) + c(|A \setminus K| + |Du|(A \setminus K))\,.$$

Since the last term in this inequality can be taken arbitrarily small, and $B \subset\subset A$, we get the desired equality in (72).

**Step 3.** We shall prove that $\overline{G}(u, \cdot)$ is a subadditive set function on $A(\Omega)$. By the regularity of $\overline{G}(u, \cdot)$ (Step 2) it is sufficient to prove that

$$\overline{G}(u, A) \le \overline{G}(u, A_1) + \overline{G}(u, A_2)$$

for every open set $A \subset\subset A_1 \cup A_2$, with $A_1$ and $A_2$ in $A(\Omega)$. This inequality can be proved by arguing as in Step 2, choosing $K = \overline{A} \setminus A_2$ and

$$B = \{x \in \mathbb{R}^n : d_K(x) < \frac{1}{2}\mathrm{dist}(K, A \setminus A_1)\}\,.$$

For a detailed proof see Appendix C.

**Step 4.** $\overline{G}(u,\cdot)$ is a superadditive on $A(\Omega)$. This comes out easily.
By Step 1-Step 4 we get, applying Theorem 24 by De Giorgi and Letta, the statement of Proposition 5. □

### 7.3 Integral representation

**Proposition 6.** *There exists a quasiconvex function* $\psi : \mathbb{M}^{k\times n} \to [\,0,+\infty[$ *such that*

$$\overline{G}(u,A) = \int_A \psi(\nabla u(x))\,dx \tag{75}$$

*for every* $A \in A(\Omega)$ *and* $u \in W^{1,1}(\Omega;\mathbb{R}^k)$*. Furthermore, the function* $\psi$ *satisfies*

$$0 \le \psi(\xi) \le c(1+|\xi|) \qquad \forall \xi \in \mathbb{M}^{k\times n}\,. \tag{76}$$

*Proof.* We want to apply the integral representation theorem 22. The linear growth condition for $\overline{G}(u,\cdot)$ is shown in (71). The locality property follows from the definition of $\overline{G}$, while the lower semicontinuity condition is satisfied since $\overline{G}(\cdot,A)$ is $L^1$-lower semicontinuous. Finally, it is easy to check that $G$ satisfies

$$G(u+z,A) = G(u,A) \qquad \forall u \in W^{1,1}(\Omega;\mathbb{R}^k), A \in A(\Omega)\,, z \in \mathbb{R}^k\,,$$

and so does $\overline{G}$. Finally, by Proposition 5 we have condition (v) for $\overline{G}(u,\cdot)$. Therefore, there exists a quasiconvex function $\psi : \Omega \times \mathbb{M}^{k\times n} \to [\,0,+\infty[$ such that

$$\overline{G}(u,A) = \int_A \psi(x,\nabla u(x))\,dx$$

for every $u \in W^{1,1}(\Omega;\mathbb{R}^k)$ and for every $A \in A(\Omega)$.

In order to prove that $\psi$ does not depend on $x$, we observe that by the definition of $G$ and $\overline{G}$, if we compute $\overline{G}$ on the linear function $u_\xi(x) = \xi x$ with $\xi \in \mathbb{M}^{k\times n}$, we obtain

$$\int_{B_1} \psi(x,\xi)\,dx = \int_{B_2} \psi(x,\xi)\,dx$$

on any pair of congruent balls $B_1\,, B_2 \subset \Omega$. This equality implies that $\psi(x,\xi) = \psi(y,\xi)$ at every pair of Lebesgue points of the function $\psi(\cdot,\xi)$. If we choose a dense sequence $(\xi_h) \subset \mathbb{M}^{k\times n}$, using the continuity of $\psi(x,\cdot)$ we get the existence of a set $N \subset \Omega$ with $|N| = 0$, and such that

$$\psi(x,\xi) = \psi(y,\xi)$$

for every $\xi \in \mathbb{M}^{k\times n}$ and for every $x\,, y \in \Omega \setminus N$. Hence it is not restrictive to suppose

$$\psi(x,\xi) = \psi(\xi)$$

and we get (75). The inequalities in (76) follow from the integral representation (75), using (71) and the positivity of $\overline{G}$. □

### 7.4 Proof of the main theorem

Before we conclude with the proof of the relaxation theorem 21 let us give some estimates of the function $\psi$ obtained above.

**Proposition 7.** *Let $\psi$ be the function of Proposition 6. Then*

$$\psi(\xi) \le f(\xi) \qquad \forall \xi \in \mathbb{M}^{k\times n}\,, \tag{77}$$

*and*

$$\psi^\infty(a\otimes\nu) \le g(a\otimes\nu) \qquad \forall a \in \mathbb{R}^k\,, \forall \nu \in S^{n-1}\,. \tag{78}$$

**Note.** By the homogeneity of $\psi^\infty$ and $g$ we get immediately (78) for all rank-one matrices.

*Proof.* Let $u(x) = \xi x$ for a fixed $\xi \in \mathbb{M}^{k\times n}$. Then, by (75)

$$|\Omega|\psi(\xi) = \overline{G}(\xi x, \Omega) \le G(\xi x, \Omega) = |\Omega| f(\xi)\,,$$

which proves (77).

As for (78) let us fix $a \in \mathbb{R}^k$ and $\nu \in S^{n-1}$. For every $t > 0$, we can consider the linear function

$$u_t(x) = ta\langle x, \nu\rangle\,,$$

so that we have $Du_t = ta\otimes\nu$. We can approximate $u_t$ in $L^1(\Omega;\mathbb{R}^k)$ with a sequence $(u_t^h)$ in $SBV(\Omega;\mathbb{R}^k)$ defined by

$$u_t^h(x) = \frac{1}{h} ta\big[h\langle x,\nu\rangle\big]$$

which has jumps of size $1/h$ in the direction $a$, along hyperplanes orthogonal to $\nu$ at a regular distance of $1/h$, and $\nabla u_t^h = 0$ a.e. ($[s]$ is the integer part of $s$). Let $Q_\nu$ be any cube contained in $\Omega$ with an edge parallel to $\nu$; so we get

$$((u_t^h)^+ - (u_t^h)^-)\otimes \nu_{u_t^h} H^{n-1} \llcorner S_{u_t^h}\cap Q_\nu = \frac{1}{h} ta\otimes\nu H^{n-1}\llcorner S_{u_t^h}\cap Q_\nu\,.$$

We have then (using the homogeneity of $g$)

$$\begin{aligned} G(u_t^h, Q_\nu) &= f(0)|Q_\nu| + g\big(\tfrac{1}{h} ta\otimes\nu\big) H^{n-1}(S_{u_t^h}\cap Q_\nu) \\ &\le f(0)|Q_\nu| + tg(a\otimes\nu)|Q_\nu|\,. \end{aligned}$$

Hence we obtain

$$\psi(ta\otimes\nu) = \frac{1}{|Q_\nu|}\overline{G}(u_t, Q_\nu) \le \frac{1}{|Q_\nu|}\liminf_{h\to+\infty} G(u_t^h, Q_\nu)$$

$$\le f(0) + tg(a\otimes\nu)\,.$$

Dividing by $t$, and letting $t\to+\infty$, we obtain

$$\psi^\infty(a\otimes\nu) \le g(a\otimes\nu)\,,$$

which is inequality (78). □

*Proof of the relaxation theorem 21.*

Let $H : BV(\Omega;\mathbb{R}^k) \to [\,0,+\infty\,]$ be the functional on the right-hand side of (59), i.e.,

$$H(u) = \int_\Omega \varphi(\nabla u(x))\,dx + \int_\Omega \varphi^\infty\big(\frac{D_s u}{|D_s u|}\big)\,|D_s u|\,, \tag{79}$$

where the function $\varphi : \mathbb{M}^{k\times n} \to [\,0,+\infty[$ is given by

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex}, \psi \le f \text{ on } \mathbb{M}^{k\times n}, \psi^\infty \le g \text{ on } \mathbb{M}_1^{k\times n}\}\,.$$

**Step 1.** $H \le \overline{F}$ on $BV(\Omega;\mathbb{R}^k)$.

By the definition of $\varphi$ we get immediately that

$$0 \le \varphi(\xi) \le f(\xi) \qquad \forall \xi \in \mathbb{M}^{k\times n}\,;$$

moreover, it is easy to see that

$$\varphi^\infty \le g \qquad \text{on } \mathbb{M}_1^{k\times n}\,.$$

Therefore $H \le F$ on $BV(\Omega;\mathbb{R}^k)$. If we show that $H$ is $L^1$-lower semicontinuous on $BV(\Omega;\mathbb{R}^k)$ we get $H \le \overline{F}$ on $BV(\Omega;\mathbb{R}^k)$. We will apply the semicontinuity theorem 23 by Ambrosio and Dal Maso; we have to prove that

$$0 \le \varphi(\xi) \le c(1+|\xi|) \qquad \forall \xi \in \mathbb{M}^{k\times n}\,.$$

It suffices to show that

$$\phi(\xi) \le c(1+|\xi|) \qquad \forall \xi \in \mathbb{M}^{k\times n}$$

for every quasiconvex function $\phi$ such that $\phi \le f$ on $\mathbb{M}^{k\times n}$ and $\phi^\infty \le g$ on $\mathbb{M}_1^{k\times n}$. Let us fix such a $\phi$. Since $\phi$ is quasiconvex, it is rank-one convex, which implies that for every $\xi \in \mathbb{M}^{k\times n}$, $a \in \mathbb{R}^k$, $b \in \mathbb{R}^n$ the function

$$t \longmapsto \phi(\xi + ta\otimes b) \quad \text{is convex on } \mathbb{R}.$$

In particular we get (using a truncation argument) that

$$\phi(\xi + a \otimes b) \leq \phi(\xi) + \phi^\infty(a \otimes b)\,. \tag{80}$$

Since every $\xi \in \mathbb{M}^{k\times n}$ can be decomposed as $\xi = \sum_{m=1}^{n} a_m \otimes e_m$ for suitable $a_m \in \mathbb{R}^k$ ($(e_m)$ stands for the standard basis of $\mathbb{R}^n$), we get by applying repeatedly (80)

$$\begin{aligned}\phi(\xi) &= \phi(\sum_{m=1}^{n} a_m \otimes e_m) = \phi(\sum_{m=1}^{n-1} a_m \otimes e_m + a_n \otimes e_n)\\ &\leq \phi(\sum_{m=1}^{n-1} a_m \otimes e_m) + \phi^\infty(a_n \otimes e_n) \leq \phi(0) + \sum_{m=1}^{n} \phi^\infty(a_m \otimes e_m)\\ &\leq f(0) + \sum_{m=1}^{n} g(a_m \otimes e_m) \leq f(0) + M \sum_{m=1}^{n} |a_m \otimes e_m| \leq c(1+|\xi|)\,,\end{aligned}$$

where $M$ is defined by (65). Finally, $\varphi$ is quasiconvex and satisfies (62) and we conclude that $H$ is $L^1$-lower semicontinuous on $BV(\Omega;\mathbb{R}^k)$.

**Step 2.** $H \geq \overline{F}$ on $BV(\Omega;\mathbb{R}^k)$.

By the definition of $\overline{F}$, by applying the relaxation result in Theorem 23 we get for every $u \in BV(\Omega;\mathbb{R}^k)$

$$\overline{F}(u) \leq \overline{G} \leq \overline{\overline{G} + \chi_{W^{1,1}(\Omega;\mathbb{R}^k)}} = \int_\Omega \psi(\nabla u(x))\,dx + \int_\Omega \psi^\infty(\frac{D_s u}{|D_s u|})\,|D_s u|\,.$$

Applying Proposition 7 and the definition of $\varphi$ in (60) we have that $\psi \leq \varphi$ and we get

$$\overline{F}(u) \leq \int_\Omega \varphi(\nabla u(x))\,dx + \int_\Omega \varphi^\infty(\frac{D_s u}{|D_s u|})\,|D_s u| = H(u)$$

for $u \in BV(\Omega;\mathbb{R}^k)$. □

**Additional remarks**

*Remark 30. If $f$ is positively 1-homogeneous, then $\varphi$ is positively 1-homogeneous.* In fact, it is immediate to check that $\psi$ is a quasiconvex function such that $\psi \leq f$ on $\mathbb{M}^{k\times n}$ and $\psi^\infty \leq g$ on $\mathbb{M}_1^{k\times n}$ if and only if for every fixed $\lambda > 0$ such is the function $\phi(\xi) = \dfrac{1}{\lambda}\psi(\lambda\xi)$. We then obtain

$$\begin{aligned}\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex, positively 1-homogeneous, } \psi \leq f \text{ on } \mathbb{M}^{k\times n},\\ \psi^\infty \leq g \text{ on } \mathbb{M}_1^{k\times n}\}\end{aligned}$$

$$= \sup\{\psi(\xi) : \psi \text{ quasiconvex, positively 1-homogeneous, } \psi \leq f \wedge g \text{ on } \mathbb{M}^{k\times n}\}$$

(since $\psi = \psi^\infty$ for $\psi$ positively 1-homogeneous; we have extended $g = +\infty$ on $\mathbb{M}^{k\times n} \setminus \mathbb{M}_1^{k\times n}$). Anyhow, Müller has shown in [42] that quasiconvexity and positive 1-homogeneity does not imply convexity.

*Remark 31. Partitions.* Let us consider the case

$$f(\xi)\begin{cases} 0 & \text{if } \xi = 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Then the functional $F < +\infty$ only on functions in the space $SBV(\Omega;\mathbb{R}^k)$ with $\nabla u = 0$ a.e. These functions can be identified with "partitions of $\Omega$ in sets of finite perimeter" (see [9], [10], [29] ). Every such function can be expressed as

$$u = \sum_{j\in\mathbb{N}} c_j \mathbf{1}_{E_j}\,,$$

where $c_j \in \mathbb{R}^k$, and $(E_j)$ is a partition of $\Omega$ in sets of finite perimeter. The functional can then be written in the form

$$F(u) = \sum_{i,j\in\mathbb{N}} \frac{1}{2} \int_{(\partial^* E_i \cap \partial^* E_j)\cap\Omega} g((c_j - c_i)\otimes \nu_j)\, dH^{n-1}\,, \tag{81}$$

where $\nu_j$ is the interior normal to $E_j$, and $\partial^* E_j$ denotes the reduced boundary of $E_j$. Since $f$ is positively 1-homogeneous, by the previous remark, the relaxed functional $\overline{F}$ is given by

$$\overline{F}(u) = \int_\Omega \varphi(\nabla u)\, dx + \int_\Omega \varphi\big(\frac{D_s u}{|D_s u|}\big)\, |D_s u|$$

with

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex, positively 1-homogeneous, } \psi \le g \text{ on } \mathbb{M}_1^{k\times n}\}$$

($\varphi = \varphi^\infty$ since $\varphi$ is positively 1-homogeneous). Note that if in particular $g$ is quasiconvex then the functional (81) defined on partitions is lower semicontinuous with respect to the $L^1$-convergence.

## 8 Relaxation in higher dimension (isotropy assumptions)

In this section we state a relaxation result for functionals defined on $SBV(\Omega)$ with a more general surface energy density as the functionals defined in (43) but with isotropy assumptions on the bulk energy term.

Before we state the relaxation result by [17] let us introduce the notion of inf-convolution. Let $f_1,\ f_2 : \mathbb{R} \to [\,0,+\infty\,]$ be proper (that is, $f_i(x) < +\infty$ for at least on $x$) convex functions, then the *inf-convolution* of $f_1$ and $f_2$ is defined as

$$(f_1 \,\nabla\, f_2)(z) = \inf\{f_1(x) + f_2(z - x) : x \in \mathbb{R}\} \tag{82}$$

and it turns out to be a convex function (see [43]). Moreover, one can prove that if $f_1$ and $f_2$ are non-negative convex functions on $\mathbb{R}$, with $f_1(0) = 0$ and $f_2$ positively homogeneous of degree 1, then

$$(f_1 \nabla f_2)(z) = (f_1 \wedge f_2)^{**}(z) \,.$$

Moreover, if $f$ is convex, $f(0) = 0$, and $\theta$ is subadditive, positive and $\theta(0) = 0$, then

$$\begin{cases} f \nabla f^\infty = f & f \nabla \theta^0 = (f \nabla \theta^0)^{**} \\ f^\infty \nabla \theta = \overline{\mathrm{sub}}\,(f^\infty \wedge \theta) & \\ (f^\infty \nabla \theta)^0 = f^\infty \nabla \theta^0 = (f \nabla \theta^0)^\infty \,. & \end{cases} \tag{83}$$

**Theorem 25 (Relaxation in $BV(\Omega)$ - Isotropy assumptions: interaction between bulk and interfacial energy densities).** *Let $f : \mathbb{R} \to [\,0, +\infty\,]$ be a convex function with $f(0) = 0$; let $g : \mathbb{R} \to [\,0, +\infty\,]$ be lower semicontinuous with $g(0) = 0$, locally bounded and such that the map $t \mapsto g(|t|)$ is subadditive, i.e., $g(|t_1 + t_2|) \leq g(|t_1|) + g(|t_2|)$ for every $t_1, t_2 \in \mathbb{R}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $F : BV(\Omega) \to [\,0, +\infty\,]$ be defined by*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(|\nabla u|)\,dx + \int_{S_u \cap \Omega} g(|u^+ - u^-|)\;dH^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise on } BV(\Omega). \end{cases}$$

*Then the relaxed functional of $F$ with respect to the $BV - w^*$ topology on $BV(\Omega)$ can be written as*

$$\overline{F}(u) = \int_\Omega f_1(|\nabla u|)\,dx + c_1 \int_{\Omega \setminus S_u} |D_s u| + \int_{\Omega \cap S_u} g_1(|u^+ - u^-|)\,dH^{n-1} \,,$$

*where*

$$\begin{array}{ll} f_1 = f \nabla g^0 & (= (f \wedge g^0)^{**}) \\ g_1 = g \nabla f^\infty & c_1 = f_1^\infty(1) \quad (= g_1^0(1)). \end{array}$$

*Remark 32.* Let us point out that the proof of this result follows an analogous scheme as seen in Section 7 for the proof of Theorem 21. The estimate of the lower bound of the relaxed functional uses a lower semicontinuity theorem proved previously in the paper quoted above. The upper bound is obtained by using a measure theoretical approach as in Theorem 21, the relaxation result in Theorem 17 and a representation theorem on partitions (see [9]).

*Remark 33.* The authors suppose $0 < \min\{f^\infty(1), g^0(1)\} < +\infty$. Let us note that in the case $\min\{f^\infty(1), g^0(1)\} = +\infty$, we get $f_1 = f$, $g_1 = g$, $c_1 = +\infty$, and then $\overline{F} = F$. This lower semicontinuity theorem is a result proven by [5], see Theorem 12 of these lecture notes. On the other hand, if $0 = \min\{f^\infty(1), g^0(1)\}$ then $\overline{F} = 0$.

**Applications**

By applying the above theorem it follows immediately by easy calculations that with

$$f(t) = |t|^p \qquad\qquad g(t) = |t|^q \,,$$

we get that for

(a) $1 < p < +\infty$ and $0 < q < 1$

$$F(u) = \int_{\Omega} |\nabla u|^p dx + \int_{S_u \cap \Omega} |u^+ - u^-|^q \, dH^{n-1}$$

(b) $p = 1$ and $q = 1$

$$F(u) = \int_{\Omega} |\nabla u| \, dx + \int_{S_u \cap \Omega} |u^+ - u^-| \, dH^{n-1}$$

are $BV - w^*$ lower semicontinuous functionals on $SBV(\Omega)$. Furthermore, if

(c) $p > 1$ and $q = 1$

$$f_1(\xi) = (|\xi|^p \wedge |\xi|)^{**} = \begin{cases} |\xi|^p & \text{if } |\xi| \le p^{\frac{1}{1-p}} \\ |\xi| - (p-1)p^{\frac{p}{1-p}} & \text{if } |\xi| > p^{\frac{1}{1-p}} \end{cases}$$

$$g_1(\xi) = |\xi| \qquad c_1 = 1\,,$$

(d) $p = 1$ and $0 < q < 1$

$$f_1(\xi) = |\xi|$$

$$g_1(\xi) = \begin{cases} |\xi| & \text{if } |\xi| \le 1 \\ |\xi|^q & \text{if } |\xi| \ge 1 \end{cases} \qquad c_1 = 1\,.$$

For further applications see [26].

*Remark 34.* (see [26]) Some interesting examples do not fall into the known framework. Consider for instance a bulk energy density $f(z)$ with growth $p > 1$ (as in the case of linear elasticity, where $p = 2$) and a fracture energy density $g$ of the form

$$g(s, \nu) = \begin{cases} 1 & \text{if } \langle s, \nu \rangle > 0 \\ +\infty & \text{if } \langle s, \nu \rangle < 0 \\ \gamma(s) & \text{if } \langle s, \nu \rangle = 0, \end{cases}$$

where $\gamma$ is a function with linear growth. The term 1 describes the energy necessary to produce a normal detachment according to the Griffith model, the term $+\infty$ is to avoid the interpenetration of the two sides of the fracture, and the term $\gamma(s)$ takes into account the tangential sliding. It would be interesting to compute the relaxed functional associated to the model above, and to see if the formulas for $f_1$ and $g_1$ obtained in Theorem 25 still hold.

Let us mention that in [24] we consider the case when also normal detachments are forbidden (i.e., $g = +\infty$ if $\langle s, \nu \rangle \ne 0$), taking

$$f(z) = \frac{1}{2}|e^D(z)|^2 + \alpha |\mathrm{tr}(z)|^2\,, \qquad g(s, \nu) = \begin{cases} \dfrac{\beta |s|}{\sqrt{2}} & \text{if } \langle s, \nu \rangle = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $e^D(z)$ is the deviator of the symmetric part of $z$ defined by $e^D(z) = \dfrac{z+z^t}{2} - \dfrac{\mathrm{tr}(z)}{n} I$. We obtained the relaxed integrands

$$\overline{f}(z) = \varphi(e^D(z)) + \alpha|\mathrm{tr}(z)|^2 \qquad \overline{g}(s,\nu) = \varphi^\infty(e^D(s \otimes \nu)),$$

where $\varphi$ is the conjugate function on the set of all symmetric matrices with null trace of the function

$$h(\sigma) = \begin{cases} \frac{1}{2}\sigma^2 & \text{if } \lambda_M(\sigma) - \lambda_m(\sigma) \le \sqrt{\beta}, \\ +\infty & \text{otherwise} \end{cases}$$

being $\lambda_M(\sigma)$, $\lambda_m(\sigma)$ the maximum and minimum eigenvalue, respectively, of $\sigma$.

In this setting let us also quote the paper by [25] on relaxation of elastic energies with free discontinuities and constraint on the strain.

Finally I will mention the paper by [18], where the authors introduce a new method for the identification of the integral representation of functionals defined on $BV(\Omega;\mathbb{R}^k) \times A(\Omega)$. Some relaxation results in $SBV(\Omega;\mathbb{R}^k)$ are recovered in the case where bulk and surface energies are present.

# 9 Appendix

## 9.1 Appendix A: Lower semicontinuous envelope

### Subadditive envelope

**Definition.** The *subadditive envelope* $\mathrm{sub}\,\varphi$ of $\varphi : \mathbb{R} \to \mathbb{R}$ is the greatest subadditive function less than or equal $\varphi$.

It is easy to see that

$$\mathrm{sub}\,\varphi(x) = \inf\Big\{\sum_{k=1}^m \varphi(x_k) : \sum_{k=1}^m x_k = x\,, m = 1,2,\dots\Big\}.$$

**Definition.** $\overline{\mathrm{sub}}\,\varphi(x) = \mathrm{sub}\,(\mathrm{sc}^-\varphi)(x)$ for every $x \in \mathbb{R}$.

**Note.** If $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\phi(x,y) = \varphi(x-y)$ then $\overline{\mathrm{sub}}\,\phi(x,y) = \overline{\mathrm{sub}}\,\varphi(x-y)$.

A1. If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex (hence continuous), then $\overline{\mathrm{sub}}\,\varphi = \mathrm{sub}\,\varphi$ can be computed more easily:

$$\mathrm{sub}\,\varphi(x) = \inf\Big\{k\varphi\Big(\frac{x}{k}\Big) : k = 1,2,\dots\Big\}.$$

This follows immediately by the convexity inequality $k\varphi\big(\frac{x}{k}\big) \le \sum_{j=1}^k \varphi(y_j)$, whenever $y = \sum_{j=1}^k y_j$.

A2. If $\varphi : \mathbb{R} \to \mathbb{R}$ is subadditive and locally bounded, then it grows less than linearly at infinity. Indeed, for every $y \in \mathbb{R}$ we have by subadditivity

$$\varphi(y) = \varphi\big((1+[|y|])\frac{y}{(1+[|y|])}\big) \leq (1+[|y|])\varphi\big(\frac{y}{(1+[|y|])}\big)\,,$$

where $[s]$ is the integer part of $s$. Since $M = \sup_{|x|\leq 1} |\varphi(x)| < +\infty$ we get

$$\varphi(y) \leq M(1+|y|) \qquad \forall y \in \mathbb{R}\,.$$

A3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(x) = 1 + x^2$. By A2 we get that $\varphi$ is not subadditive. Since $\varphi$ is convex, by A1 we have

$$\operatorname{sub}\varphi(x) = \min\big\{k + \frac{t^2}{k} : k = 1, 2, \ldots\big\}\,.$$

A4. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(x) = (2|x| - 1) \vee 1$. Since $\varphi$ is convex and even, by A1 we get that $\operatorname{sub}\varphi$ is even and continuous and in $[\,0, +\infty[$ we have

$$\operatorname{sub}\varphi(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ k + 2(x-k) & \text{if } k \leq x \leq k + \frac{1}{2} \qquad k = 1, 2, \ldots \\ k+1 & \text{if } k + \frac{1}{2} \leq x \leq k+1 \qquad k = 1, 2, \ldots \end{cases}$$

In this case we have

$$|x| \leq \operatorname{sub}\varphi(x) \leq |x| + \frac{1}{2}$$

for $|x| \geq \frac{1}{2}$. We have $\operatorname{sub}\varphi(x) = |x|$ for $x = \pm 1\,, \pm 2\,, \ldots$, $\operatorname{sub}\varphi(x) = |x| + \frac{1}{2}$ for $x = \pm\frac{1}{2}\,, \pm\frac{3}{2}\,, \ldots$, and hence $\operatorname{sub}\varphi$ is not asymptotic to a linear function as $x \to \pm\infty$.

A5. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(x) = |x - 1|$. By A1 we have

$$\operatorname{sub}\varphi(x) = \min\{|x-k| : k = 1, 2, \ldots\}$$

i.e.,

$$\operatorname{sub}\varphi(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ \operatorname{dist}(x, \mathbb{N}) & \text{if } x \geq 1. \end{cases}$$

In this case we have that the limit $\lim_{x\to+\infty} \operatorname{sub}\varphi(x)$ does not exist.

*Proof of Proposition 3.*

(i) Let $\lambda = \inf\phi > 0$. Then $\lambda \leq \phi(x, y)$ for every $x, y \in \mathbb{R}$. By the definition of $\mathrm{sc}^-\phi$ (= greatest lower semicontinuous function less than or equal $\phi$, see Definition 4) we get $\lambda \leq (\mathrm{sc}^-\phi)(x, y) \leq \phi(x, y)$ for all $x, y \in \mathbb{R}$. Therefore, $\lambda' = \inf(\mathrm{sc}^-\phi) \geq \lambda > 0$. Being $\psi(x, y) \equiv \lambda'$ a subadditive function less than or equal $\phi(x, y)$ for every $x, y \in \mathbb{R}$, by definition of subadditive envelope we get

$$0 < \lambda' \le \operatorname{sub}(\operatorname{sc}^-\phi)(x,y) = \overline{\operatorname{sub}}\,\phi(x,y) \le \phi(x,y)$$

for every $x, y \in \mathbb{R}$. Hence $\inf \overline{\operatorname{sub}}\,\phi > 0$. From $\phi(x,y) \ge |x-y|$ for every $x, y \in \mathbb{R}$ it follows that $|x-y| \le (\operatorname{sc}^-\phi)(x,y) \le \phi(x,y)$ for all $x, y \in \mathbb{R}$. Since $\psi(x,y) = |x-y|$ is subadditive, by definition of subadditive envelope we get

$$|x-y| \le \overline{\operatorname{sub}}\,\phi(x,y) \le \phi(x,y) \qquad \forall x, y \in \mathbb{R}\,.$$

(ii) Let us prove that $\overline{\operatorname{sub}}\,\phi$ is lower semicontinuous. Fixed $x, y \in \mathbb{R}$ and two sequences $x_h \to x$, $y_h \to y$ such that there exists the limit $\lim_{h\to+\infty} \overline{\operatorname{sub}}\,\phi(x_h, y_h)$, we have to prove that

$$\overline{\operatorname{sub}}\,\phi(x,y) \le \lim_{h\to+\infty} \overline{\operatorname{sub}}\,\phi(x_h, y_h)\,.$$

By definition for every $h \in \mathbb{N}$ there exist $x_0^h, \dots, x_{m_h}^h$ such that $x_0^h = y_h$, $x_{m_h}^h = x_h$, and

$$\sum_{k=1}^{m_h} (\operatorname{sc}^-\phi)(x_k^h, x_{k-1}^h) \le \overline{\operatorname{sub}}\,\phi(x_h, y_h) + \frac{1}{h}\,.$$

By the condition $\inf(\operatorname{sc}^-\phi) > 0$ we have that the sequence $(m_h)$ is bounded. Hence we can suppose $m_h = m$, independent of $h$. The condition $(\operatorname{sc}^-\phi)(x,y) \ge |x-y|$ implies that all sequences $(x_0^h), \dots, (x_{m_h}^h)$ are bounded. Again we can suppose then that $x_0^h \to x_0, x_1^h \to x_1, \dots, x_m^h \to x_m$ for some $x_0, x_1, \dots, x_m \in \mathbb{R}$. Of course, $x_0 = y$, and $x_m = x$. By semicontinuity of $\operatorname{sc}^-\phi$ we get

$$(\operatorname{sc}^-\phi)(x_k, x_{k-1}) \le \liminf_{h\to+\infty} (\operatorname{sc}^-\phi)(x_k^h, x_{k-1}^h)\,,$$

and hence we get

$$\begin{aligned}\sum_{k=1}^{m} (\operatorname{sc}^-\phi)(x_k, x_{k-1}) &\le \sum_{k=1}^{m} \liminf_{h\to+\infty} (\operatorname{sc}^-\phi)(x_k^h, x_{k-1}^h) \\ &\le \liminf_{h\to+\infty} \sum_{k=1}^{m} (\operatorname{sc}^-\phi)(x_k^h, x_{k-1}^h) \le \lim_{h\to+\infty} \overline{\operatorname{sub}}\,\phi(x_h, y_h)\,.\end{aligned}$$

By definition this shows that

$$\overline{\operatorname{sub}}\,\phi(x,y) \le \lim_{h\to+\infty} \overline{\operatorname{sub}}\,\phi(x_h, y_h)\,.$$

Suppose now that there exists $\phi^1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ lower semicontinuous and subadditive such that $\phi^1 \le \phi$. We get then immediately that $\phi^1 \le \operatorname{sc}^-\phi \le \phi$ (by definition of $\operatorname{sc}^-\phi$). Furthermore, being $\phi^1$ subadditive

$$\phi^1 \le \operatorname{sub}(\operatorname{sc}^-\phi) \le \phi\,.$$

This proves (ii). □

### 9.2 Appendix B: Proof of (73)

Let us show that for every $h \in \mathbb{N}$, the set $t \in\ ]0, \delta[$ that does not satisfy

$$H^{n-1}(S_{u_h} \cap \partial^* B_t) = 0$$

is at most countable. Let $T_h = \{t \in\ ]0, \delta[: H^{n-1}(S_{u_h} \cap \partial^* B_t) > \frac{1}{h}\}$. We claim that $T_h$ is finite. Indeed, if $T_h$ is countable we get

$$H^{n-1}(S_{u_h} \cap \Omega) \geq H^{n-1}\big(\bigcup_{t \in T_h} S_{u_h} \cap \partial^* B_t\big) = \sum_{t \in T_h} H^{n-1}(S_{u_h} \cap \partial^* B_t)$$
$$\geq \sum_{t \in T_h} \frac{1}{h} = +\infty\,.$$

Therefore $T_h$ has to be finite (since $H^{n-1}(S_{u_h} \cap \Omega) < +\infty$ holds). We conclude that

$$T = \{t \in\ ]0, \delta[: H^{n-1}(S_{u_h} \cap \partial^* B_t) > 0\} = \bigcup_h T_h$$

is at most countable. $\square$

### 9.3 Appendix C: Proof of Step 3 in Proposition 67

Let us show that for every $A_1,\ A_2 \in A(\Omega)$

$$\overline{G}(u, A) \leq \overline{G}(u, A_1) + \overline{G}(u, A_2)$$

for every open set $A \subset\subset A_1 \cup A_2$.

It is not restrictive to suppose $\overline{G}(u, A_i) < +\infty\,, i = 1, 2$. Fixed $A \subset\subset A_1 \cup A_2$, let us choose the compact set $K = \overline{A} \setminus A_2$, and set

$$\delta = \tfrac{1}{2}\mathrm{dist}(K, A \setminus A_1)$$

$$B_t = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) < t\} \quad \text{for } t \in\ ]0, \delta[ \qquad\qquad B = B_\delta\,.$$

Note that for a.e. $t \in\ ]0, \delta[$, $B_t$ is a set of finite perimeter. Let us consider the minimizing sequences $(u_h^i) \subset SBV(\Omega; \mathbb{R}^k$, $u_h^i \to u$ in $L^1(A_i; \mathbb{R}^k)$ with $H^{n-1}(S_{u_h^i} \cap \Omega) < +\infty$ and

$$\overline{G}(u, A_i) = \lim_{h \to +\infty} G(u_h^i, A_i) \qquad i = 1, 2\,.$$

Since we have $H^{n-1}(S_{u_h^i} \cap \Omega) < +\infty$, we get

$$H^{n-1}(S_{u_h^i} \cap \Omega \cap \partial^* B_t) = 0 \tag{84}$$

for a.e. $t \in\ ]0, \delta[$ (see Appendix B). By the Fleming-Rishel co-area formula and the mean value theorem, for every $h \in \mathbb{N}$, we can choose $t_h \in\ ]0, \delta[$ such that (84) holds, $B_{t_h}$ is a set of finite perimeter, and

$$\int_{\partial^* B_{t_h} \cap A} |\widetilde{u}_h^1 - \widetilde{u}_h^2| \, dH^{n-1} \leq \frac{1}{\delta} \int_A |u_h^1 - u_h^2| \, dx \,, \tag{85}$$

where $\widetilde{u}_h^i$ are the approximate limits of $u_h^i$ (they exist $H^{n-1}$-a.e. on $\partial^* B_{t_h}$ by (84)).

Let us define the sequence

$$w_h = \begin{cases} u_h^1 & \text{on } \Omega \cap B_{t_h} \\ u_h^2 & \text{on } \Omega \setminus B_{t_h} . \end{cases}$$

Then $w_h \in SBV(\Omega; \mathbb{R}^k)$, $H^{n-1}(S_{w_h} \cap \Omega) < +\infty$, and $w_h \to u$ in $L^1(A; \mathbb{R}^k)$. Moreover, on $A$ we have

$$\nabla w_h = \nabla u_h^1 \, \mathbf{1}_{A \cap B_{t_h}} + \nabla u_h^2 \, \mathbf{1}_{A \setminus B_{t_h}}$$

and the Hausdorff part of the measure $Dw_h$ is given by

$$((u_h^1)^+ - (u_h^1)^-) \otimes \nu_{u_h^1} H^{n-1} \llcorner S_{u_h^1} \cap A \cap B_{t_h} +$$
$$+ ((u_h^2)^+ - (u_h^2)^-) \otimes \nu_{u_h^2} H^{n-1} \llcorner S_{u_h^2} \cap (A \setminus B_{t_h}) +$$
$$+ (\widetilde{u}_h^1 - \widetilde{u}_h^2) \otimes \nu_{\partial^* B_{t_h}} H^{n-1} \llcorner \partial^* B_{t_h} \cap A \,,$$

where $\nu_{\partial^* B_{t_h}}$ is normal to $\partial^* B_{t_h}$ pointing inwards $B_{t_h}$. Then (using the positive 1-homogeneity of $G$, (65) and (85))

$$\begin{aligned} G(w_h, A) &\leq G(u_h^1, A \cap B_{t_h}) + G(u_h^2, A \setminus B_{t_h}) + \\ &\qquad + \int_{\partial^* B_{t_h} \cap A} g\big((\widetilde{u}_h^1 - \widetilde{u}_h^2) \otimes \nu_{\partial^* B_{t_h}}\big) \, dH^{n-1} \\ &\leq G(u_h^1, A_1) + G(u_h^2, A_2) + \tfrac{M}{\delta} \int_A |u_h^1 - u_h^2| \, dx \,. \end{aligned}$$

By taking the limit as $h \to +\infty$, we get

$$\overline{G}(u, A) \leq \liminf_{h \to +\infty} G(w_h, A) \leq \overline{G}(u, A_1) + \overline{G}(u, A_2) \,.$$

as desired. □

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# Convergence of Dirichlet forms on fractals

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## 1 Introduction

The purpose of these notes is to introduce the theory of the convergence, and specially, to discuss the $\Gamma$-convergence, of Dirichlet forms on fractals. A Dirichlet form is a sort of “energy”, in some sense it is a generalization of the Dirichlet integral $u \mapsto \int |\operatorname{grad} u|^2$ defined for $u$ in open regions in $\mathbb{R}^n$. The investigation of Dirichlet forms on fractals began in connection with the construction of a Brownian motion on fractals. It is also closely related to the construction of a Laplacian or in other words to the definition of harmonic functions on a fractal. In some sense, on fractals, the notions of Dirichlet forms, Brownian motion, and harmonic functions are, in fact, three different points of view of the same notion. In these Notes I will discuss the notion of a Dirichlet form, as well as that of a harmonic function, as we will need it in connection with some properties of Dirichlet forms. On the contrary, I will not discuss the notion of a Brownian motion. The class of fractals which I discuss here is that of finitely ramified fractals. More or less we are in the following situation. We are given finitely many similarities in $\mathbb{R}^\nu$ with $\nu \geq 1$, which we denote by $\psi_1, ..., \psi_k$. Suppose they are contractions, i.e., their factors $r_1, ..., r_k$ are $< 1$. Then, there exists a unique nonempty compact $K$ in $\mathbb{R}^\nu$ such that $K = \bigcup_{i=1}^{k} \psi_i(K)$ (see [4]). We will say that such a set $K$ is the fractal generated by the set of similarities. The finite ramification means, more or less, that different copies $\psi_i(K)$ of $K$ can have at most finitely many (in general at most one) common points. This class of fractals includes for example the Gasket, but not the Carpet. The construction of a Dirichlet form on fractals that are not finitely ramified is much more complicated and will be not discussed in these Notes. A finitely ramified fractal can also be seen as the closure of the union of an increasing sequence of finite sets $V^{(0)}, ..., V^{(n)}, ....$ In defining a Dirichlet form on the fractal, we meet the problem that usually, $K$ has an empty interior, hence we cannot define the gradient on it, at least in the usual sense. So, we construct a Dirichlet form on $K$, by taking the limit of discrete

Dirichlet forms on $V^{(n)}$. A nontrivial point in this construction is the existence of such a limit form. However, if the initial form $E$ on $V^{(0)}$ is an eigenvector of a particular nonlinear minimization operator, also called renormalization, in our terms if $E$ is an eigenform, then we can define a sequence of discrete Dirichlet forms $E^{\Sigma}_{(n)}$, that are in some sense the sum of the copies of $E$ on the $n$-cells, suitably renormalized. Now, for every function $v$ defined on $K$, the sequence $E^{\Sigma}_{(n)}(v)$ is increasing and thus has a limit which we will denote by $E^{\Sigma}_{(\infty)}(v)$. Such a limit form turns out to be a Dirichlet form on $K$. Sections 2 to 4 are introductory and recall the construction and the main properties of a Dirichlet form on finitely ramified fractals. For the general theory of Dirichlet forms the reader can refer to [2]. The construction of Dirichlet forms on finitely ramified fractals is described for example in [1], [8], which specially stress the probabilistic point of view, and in [6] which instead deals with Dirichlet forms and Laplacian, in a very general class of fractals, called p.c.f. self-similar sets, introduced by J. Kigami in [5]. The approach followed here is, in some sense, similar to that in [6], but, in order to simplify the presentation, I have preferred not to give the general definition of Dirichlet form, but only to discuss a particular kind of Dirichlet form. An even more similar approach to that presented here can be found in [20]. I have preferred to start, in Section 2, with a specific example, the Sierpinski Gasket, as, in my opinion it allows us to better understand the problem of the construction of a Dirichlet form in a simple case. In Section 3, I first give a general definition of finitely ramified fractals, and then I discuss in such a class the construction of a Dirichlet form. While in the case of Gasket, by symmetry, we explicitly found an eigenform, in the general case of finitely ramified fractals, the existence of an eigenform is a nontrivial problem, and there are finitely ramified fractals having no eigenforms. However, the class of finitely ramified fractals having at least an eigenform includes several important examples of finitely ramified fractals, in particular, the nested fractals introduced by T. Lindstrøm [9], and we will only restrict our considerations to such a class of fractals. Section 4 is devoted to the investigation of the properties of the renormalization operator and of the related notion of the harmonic extension. The actual aim of these Notes, i.e., the convergence of Dirichlet forms on finitely ramified fractals, is described in Sections 5 and 6. While Sections 1 to 4 mainly describe known results by other authors, Sections 5 and 6 mainly treat results of the author, specially in [16]. When $E$ is not an eigenform we cannot use the same process as for the eigenforms to construct a Dirichlet form starting with $E$, since the corresponding sequence of discrete forms is no longer increasing. It could be proved that nevertheless, the discrete forms pointwise converge to a Dirichlet form defined on $K$, but the proof is more complicated (see [17]), and will be not discussed here. I will discuss, instead, the $\Gamma$-convergence of the sequence of discrete forms. I will show that in fact $E^{\Sigma}_{(n)}$ $\Gamma$-converges to the Dirichlet form on $K$ associated to an eigenform $\tilde{E}$. We will obtain $\tilde{E}$ as the limit of a sequence of forms $E_{(n)}$ on $V^{(0)}$, defined as $\tilde{M}_1^n(E)$ where $\tilde{M}_1$ is the renormalization operator divided by the eigenvalue $\rho$. The convergence of $\tilde{M}_1^n(E)$ is the most delicate point in the proof of $\Gamma$-convergence. It is, in some sense, a problem of convergence of the iterated of a map that it is known to have a fixed point. In the present case, the fixed point is the eigenform. In Section 5, I discuss the $\Gamma$-convergence on the Gasket. Due to the symmetry of the Gasket, the proof of the convergence of $\tilde{M}_1^n(E)$ is much simpler than in the general case. There are many different proofs of this result in the case of the Gasket. The proof presented here is probably not the simplest, but suggests the way of proceeding in the general case. Then, following an argument due to S. Kozlov [7], we deduce the $\Gamma$-convergence result. In Section 6, following [16], I prove the convergence of $\tilde{M}_1^n(E)$ in the general case. I restrict, in fact, the class of fractals in order to simplify the argument, avoiding some technical difficulties. At the end of the section, I merely hint the idea in the most general case. The class of fractals considered in the actual proof in Section 6, however, includes most of the usual fractals, thus it is sufficiently general for many purposes. The idea of the proof consists in proving that, with respect to Hilbert's projective metric, the iterated $\tilde{M}_1^n(E)$ get closer and closer. Note that in general $\tilde{M}_1$ is not a contraction and in fact can have different fixed points. Once the convergence of $\tilde{M}_1^n(E)$ is proved, the $\Gamma$-convergence result follows as in the case of the Gasket. As these Notes are not intended for specialists in fractals, I have tried to highlight the general ideas rather than the details. So, in some cases, for example for the notion of finitely ramified fractals, I have not used the most general definitions. For the same reason, I have preferred to give proofs which are not necessarily the shortest, but which require no non usually known results, like as the general theory of Hilbert's projective metric, in order to make these Notes as self-contained as possible.

I now fix some notation for the following. When we are in $\mathbb{R}^n$, we will denote by $d$ the euclidean distance, and unless specified otherwise, we will denote by $\|\ \|$ the euclidean norm. Sometimes, we will use $\mathbb{R}^A$ where $A$ is a set. In such a case, of course, on $\mathbb{R}^A$, we put the norm and the metric, obtained by identifying $\mathbb{R}^A$ with $\mathbb{R}^M$ where $M = \#A$. If $f$ is a map from a set $X$ into itself, we will denote by $f^n$ the $n^{th}$-iterated of $f$, i.e., the composition of $n$ maps equal to $f$. If $X$ is a topological space, we will denote by $C(X)$ the set of the continuous functions from $X$ into $\mathbb{R}$. If $A$ is a subset of a euclidean space, we will denote by co$A$ the convex hull of $A$.

## 2 Construction of an energy on the Gasket

The first step in our work consists in constructing an "energy" on the fractals. Energy here, means an analog of the Dirichlet integral, $u \mapsto \int |\mathrm{grad}\, u|^2$, on fractals. Probably, the reader has at least a rough idea of the notion of fractal. It is in some sense, a set that contains copies of itself, on arbitrarily small scales. In this section, in order to make these notions clear, I prefer to describe a specific example of fractal, the (Sierpinski) Gasket. In the next section I will give a precise notion of a fractal (or self-similar set). The Gasket, roughly speaking, can be constructed like the Cantor set, but starting from a triangle, instead of from a segment-line. More precisely, start with an equilateral triangle $T$, whose vertices are denoted by $P_1, P_2, P_3$, and consider the three rotation-free similarities $\psi_i$, $i = 1, 2, 3$, in $\mathbb{R}^2$, that are contractions with factor $\frac{1}{2}$ and have $P_i$ as fixed points, in formula $\psi_i(x) = P_i + \frac{1}{2}(x - P_i)$. Then the *(Sierpinski) Gasket* is the set $K$ defined by

$$K_0 = T,\ K_{n+1} = \bigcup_{i=1}^{3} \psi_i(K_n), \quad K = \bigcap_{n=0}^{\infty} K_n\,, \tag{1}$$

in other words, we split the initial triangle $T$ into four similar triangles and remove the central one. Then, we remove the central triangle in each of the three remaining triangles. Then, we repeat the same process on each of the nine remaining triangles, and so on. In Figure 1, we depict $K_0$, $K_1$ and $K_2$.

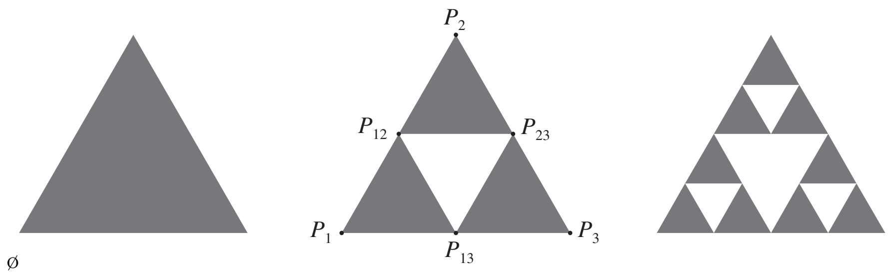


**Fig. 1.** The Sierpinski Gasket

In the previous construction, we can fix the vertices, e.g., $P_1 = (0, 0)$, $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $P_3 = (1, 0)$, in order to define a precise triangle. Put now

$$V = V^{(0)} = \{P_1, P_2, P_3\}, \quad V_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_n}(V^{(0)})$$

for $i_1, ..., i_n = 1, 2, 3$, where $\psi_{i_1,\dots,i_n}$ is an abbreviation for $\psi_{i_1} \circ ... \circ \psi_{i_n}$, and put

$$V^{(n)} = \bigcup_{i_1,..,i_n=1}^{3} V_{i_1,\dots,i_n}, \quad V^{(\infty)} = \bigcup_{n=0}^{\infty} V^{(n)}\,.$$

The sets $V_{i_1,\dots,i_n}$ are called $n$-cells and are, in some sense, small copies of $V^{(0)}$, more precisely, the copies of $V^{(0)}$ at the $n^{th}$ step. They can, of course, also be interpreted as the sets of the vertices of the triangles $\psi_{i_1,\dots,i_n}(T)$, which are copies of $T$ at the $n^{th}$ step. More generally, put $A_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_n}(A)$ for every $A \subseteq \mathbb{R}^2$. I will call $A_{i_1,\dots,i_n}$ an ($n^{th} A$) copy. Clearly, we have

$$V \subseteq K \subseteq T\,.$$

Now, we want to define an "energy" on $K$. In this section, I will give no precise meaning to the word *energy*, and I will define it in the next section, when I treat the case of general fractals. Roughly speaking, we have to define an object that in some sense resembles the Dirichlet integral. However as the Gasket has an empty interior, it is not possible to define the gradient on it. The way of constructing an energy is based on a finite-difference scheme that I will now illustrate. Since it is easy to see that $V^{(n)} \subseteq V^{(n+1)}$, and, as we will see in the following, $K$ is the closure of $V^{(\infty)}$, the idea consists in defining an energy first on $V^{(0)}$, next on $V^{(n)}$ as the sum of the copies of it on all $n$-cells, and finally on all of $K$, taking a sort of limit. More precisely, let

$$E(u) = \sum_{1\le j_1<j_2\le 3} (u(P_{j_1}) - u(P_{j_2}))^2$$

for all $u : V^{(0)} \to \mathbb{R}$. Note that $E(u) \geq 0$, and the equality holds if and only if $u$ is constant on $V^{(0)}$. This is a trivial but crucial remark. Let

$$S_0(E) = E$$

$$S_n(E)(v) = \sum_{i_1,\dots,i_n=1}^{3} E(v \circ \psi_{i_1,\dots,i_n}) \;\text{ for } v \in \mathbb{R}^{V^{(n)}}, n \geq 1\,.$$

We can also write

$$S_n(E)(v) = \sum \big(v(Q) - v(Q')\big)^2 \;\text{ for } v \in \mathbb{R}^{V^{(n)}}, n \geq 1,$$

where the sum is extended over all pairs $(Q, Q') \in V^{(n)} \times V^{(n)}$ which are *close* in the sense that they lie in a common $n$-cell. Now, we would like to define an energy $\mathcal{E}$ for functions defined on $K$ as the limit of $S_n(E)$, but some difficulties arise. First, this limit may not exist; moreover, it is 0 for too many functions, for example it is 0 on all the linear functions; we on the contrary require that it is 0 only for the constant functions. We are so lead to introduce a renormalization factor $\rho$, i.e., we take the limit of $\frac{1}{\rho^n} S_n(E)$. This is a natural device in the sense that in classical cases, such as e.g., the Dirichlet integral we have to introduce a renormalization factor. In the present case, however, the value of $\rho$ is not much expected. As we will see later, we have, in fact, $\rho = \frac{3}{5}$. In order to find the value of $\rho$, we now introduce a minimization operator. Namely, for $u : \mathbb{R}^{V^{(0)}} \to \mathbb{R}$, let

$$M_n(E)(u) = \inf\{S_n(E)(v) : v \in \mathcal{L}(n, u)\} \;\forall\, u \in \mathbb{R}^{V^{(0)}} \tag{2}$$

where $\mathcal{L}(n, u) = \{v \in \mathbb{R}^{V^{(n)}} : v = u \text{ on } V^{(0)}\}$. We will see in the following that the infimum in (2) is in fact a minimum. In terms of potential theory, $M_n(E)$ is the *trace* of $S_n(E)$ on $V^{(0)}$. For the moment let us study only $M_1$.

Put $P_{12} = \psi_1(P_2) = \psi_2(P_1)$, $P_{13} = \psi_1(P_3) = \psi_3(P_1)$, $P_{23} = \psi_2(P_3) = \psi_3(P_2)$. We thus have

$$\begin{aligned} S_1(E)(v) &= (v(P_1) - v(P_{12}))^2 + (v(P_1) - v(P_{13}))^2 + (v(P_{12}) - v(P_{13}))^2 \\ &+ (v(P_2) - v(P_{12}))^2 + (v(P_2) - v(P_{23}))^2 + (v(P_{12}) - v(P_{23}))^2 \\ &+ (v(P_3) - v(P_{13}))^2 + (v(P_3) - v(P_{23}))^2 + (v(P_{13}) - v(P_{23}))^2 \end{aligned}$$

and $M_1(E)(u)$ is the infimum of

$$\begin{gathered} (u(P_1) - x)^2 + (u(P_1) - y)^2 + (x - y)^2 \\ +(u(P_2) - x)^2 + (u(P_2) - z)^2 + (x - z)^2 \\ +(u(P_3) - y)^2 + (u(P_3) - z)^2 + (y - z)^2 \end{gathered}$$

for $x, y, z \in \mathbb{R}$. As the function to minimize is convex, it attains its minimum at the points at which its gradient is 0. Hence, $(x, y, z)$ is a minimum point if and only if

$$\begin{cases} 4x = y + z + u(P_1) + u(P_2) \\ 4y = x + z + u(P_1) + u(P_3) \\ 4z = x + y + u(P_2) + u(P_3) \end{cases} \tag{3}$$

A simple calculation yields

$$\begin{cases} x = \dfrac{2u(P_1) + 2u(P_2) + u(P_3)}{5} \\[2ex] y = \dfrac{2u(P_1) + 2u(P_3) + u(P_2)}{5} \\[2ex] z = \dfrac{2u(P_2) + 2u(P_3) + u(P_1)}{5}. \end{cases}$$

We will denote by $H_{(1;E)}(u) : V^{(1)} \to \mathbb{R}$ the so obtained solution of (3), i.e., the unique function $v \in \mathcal{L}(1, u)$ such that $M_1(E)(u) = S_1(E)(v)$. By substituting the value of $v = H_{(1;E)}(u)$, we get

$$M_1(E)(u) = \frac{3}{5} E(u),$$

thus $M_1(E)$ is a multiple of $E$. Is this a lucky case or not? To answer this question, observe that $H_{(1;E)}$ is linear, so $M_1(E)(u)$ is quadratic in $u$, in other words it is a linear combination of terms of the form $u(P_j)^2$ and of terms of the form $u(P_{j_1})u(P_{j_2})$. Moreover, since, clearly $H_{(1;E)}(u + c) = H_{(1;E)}(u) + c$, we can replace $u(P_j)$ by $u(P_j) - u(P_1)$, and thus we have that

$$M_1(E)(u) =$$
$$a(u(P_2) - u(P_1))^2 + b(u(P_3) - u(P_1))^2 + d(u(P_2) - u(P_1))(u(P_3) - u(P_1));$$

now, by the well-known formula $\alpha\beta = \frac{1}{2}(\alpha^2 + \beta^2 - (\alpha - \beta)^2)$ with $\alpha = u(P_2) - u(P_1)$, $\beta = u(P_3) - u(P_1)$, we have

$$M_1(E)(u) = a'(u(P_2) - u(P_1))^2 + b'(u(P_3) - u(P_1))^2 + d'(u(P_2) - u(P_3))^2$$

for some $a', b', d'$ and by the symmetry of the Gasket we must have $a' = b' = d'$ (I here do not prove completely the last assertion, but a formal proof of it due to symmetry can be easily given). In conclusion, there is a natural reason for which $M_1(E)$ is a multiple of $E$, and the previous calculation can be used to the only aim of evaluating the factor $\frac{3}{5}$. Note that, by the definition of $M_1(E)$, we have

$$S_1(E)(v) \geq M_1(E)(v)$$

for every $v : V^{(1)} \to \mathbb{R}$ where we identify a function $v : V^{(1)} \to \mathbb{R}$ with its restriction on $V^{(0)}$, and we will do similarly in other cases. Hence, putting $\rho = \frac{3}{5}$ we have

$$\frac{S_1(E)(v)}{\rho} \geq \frac{M_1(E)(v)}{\rho} = E(v) = \frac{S_0(E)(v)}{\rho^0}. \tag{4}$$

Suppose now $v : V^{(\infty)} \to \mathbb{R}$. We are going to prove that, more generally, the sequence $\frac{S_n(E)(v)}{\rho^n}$ is increasing in $n$, thus it has a limit, which will be the energy on $K$. To see this, note that

$$\frac{1}{\rho^{n+1}} S_{n+1}(E)(v) = \frac{1}{\rho^n} \sum_{i_1,\dots,i_n=1}^{3} \Big(\frac{1}{\rho} \sum_{i_{n+1}=1}^{3} E(v \circ \psi_{i_1,\dots,i_n,i_{n+1}})\Big)$$
$$= \frac{1}{\rho^n} \sum_{i_1,\dots,i_n=1}^{3} \Big(\frac{1}{\rho} S_1(E)(v \circ \psi_{i_1,\dots,i_n})\Big)$$
$$\geq \frac{1}{\rho^n} \sum_{i_1,\dots,i_n=1}^{3} E(v \circ \psi_{i_1,\dots,i_n}) = \frac{1}{\rho^n} S_n(E)(v)\,.$$

In the above inequality we have used (4). Now put

$$E^{\Sigma}_{(n)}(v) = \frac{1}{\rho^n} S_n(E)(v) \ \text{ for } v \in \mathbb{R}^{V^{(n)}}, \quad E^{\Sigma}_{(\infty)}(v) = \lim_{n\to\infty} E^{\Sigma}_{(n)}(v) \ \text{ for } v \in \mathbb{R}^K.$$

We have thus constructed an energy on $K$. Note that of course $E^{\Sigma}_{(\infty)}(v)$ can amount to $\infty$. An unexpected fact is that the linear nonconstant functions have infinite energy. In fact, let $v : K \to \mathbb{R}$ be linear (or more generally affine). We easily get that $E(v \circ \psi_{i_1,\ldots,i_n}) = (\frac{1}{4})^n E(v)$, hence $S_n(E)(v) = (\frac{3}{4})^n E(v)$, and $E^{\Sigma}_{(n)}(v) = (\frac{5}{4})^n E(v)$. It follows that $E^{\Sigma}_{(\infty)}(v) = +\infty$ unless $E(v) = 0$, i.e., $v$ is constant. Thus the set of the functions with finite energy is in some sense completely different from that in the case of a region in $\mathbb{R}^n$. So, one could even suspect that $E^{\Sigma}_{(\infty)}$ is identically $+\infty$. In the rest of this section we prove that this is not so. More precisely, we will prove the following two properties of $E^{\Sigma}_{(\infty)}$.

a) $E^{\Sigma}_{(\infty)}(v) = 0$ if and only if $v$ is constant on $K$.

b) The set of $v \in C(K)$ such that $E^{\Sigma}_{(\infty)}(v) < +\infty$ is dense in $C(K)$.

In order to prove a), we need a Lemma.

**Lemma 1.** *If $v : V^{(n)} \to \mathbb{R}$ and $E^{\Sigma}_{(n)}(v) = 0$, then $v$ is constant on $V^{(n)}$.*

*Proof.* By the definition of $V^{(n)}$, we have $V^{(m+1)} = \bigcup_{i=1}^{3} \psi_i(V^{(m)})$ for all $m \in \mathbb{N}$. Moreover, $\psi_i(V^{(m)}) \cap \psi_j(V^{(m)}) \supseteq \psi_i(V^{(0)}) \cap \psi_j(V^{(0)}) \neq \emptyset$. Thus, if $n > 0$ and $v$ is nonconstant on $V^{(n)}$, it is also nonconstant on $\psi_i(V^{(n-1)})$, or in other words, $v \circ \psi_i$ is nonconstant on $V^{(n-1)}$, for some $i = 1, 2, 3$, and by a recursive argument, $v \circ \psi_{i_1,i_2,\ldots,i_n}$ is nonconstant on $V^{(0)}$ for some $i_1, i_2, \ldots, i_n = 1, 2, 3$. Hence by the definition of $E^{\Sigma}_{(n)}$, we have $E^{\Sigma}_{(n)}(v) > 0$. □

**Theorem 1.** *If $v \in C(K)$ and $E^{\Sigma}_{(\infty)}(v) = 0$, then $v$ is constant on $K$.*

*Proof.* We deduce from the hypothesis, $E^{\Sigma}_{(n)}(v) = 0$ for each $n \in \mathbb{N}$, hence $v$ is constant on $V^{(\infty)}$ by Lemma 1. As $K = \overline{V^{(\infty)}}$, $v$ is constant on $K$. □

We are now going to prove b). To do this, it is useful to consider the notion of the harmonic extension on $V^{(1)}$. Given $u : V^{(0)} \to \mathbb{R}$ we know that the function $v := H_{(1;E)}(u)$ can be characterized as the unique solution of the system (3) where $v(P_{12}) = x, v(P_{13}) = y, v(P_{23}) = z$. Such a function $v$ is called *the harmonic extension* of $u$ on $V^{(1)}$. Note that (3) says that the value of $v$ at a point $Q$ of $V^{(1)} \setminus V^{(0)}$ is the mean value of the values at the points of $V^{(1)}$ close to $Q$; this is an analogous property to the mean property of harmonic functions in regions in $\mathbb{R}^n$. Another analogy is that harmonic functions also are a sort of minimum of the Dirichlet integral. Note that we have

$$E^{\Sigma}_{(1)}(H_{(1;E)}(u)) = E(u) \tag{5}$$

We will use the harmonic extension to construct a function $v$ defined on all $K$, by extending harmonically $u$ on $V^{(1)}$, next applying this process on every 1-cell to obtain a function defined on $V^{(2)}$, next applying the same process on every 2-cell and so on. However, in order to know that in this way we have in fact defined a continuous function, we have to prove that the oscillation on

$n$-cells tends to 0. The harmonic extension satisfies the following maximum principle:

**Lemma 2.** *For every* $u \in \mathbb{R}^{V^{(0)}}$ *we have*

$$\min_{V^{(0)}} u \leq H_{(1;E)}(u)(Q) \leq \max_{V^{(0)}} u \quad \forall Q \in V^{(1)} \setminus V^{(0)}$$

*(weak maximum principle). Moreover, the inequalities are strict unless* $u$ *is constant (strong maximum principle).*

*Proof.* It suffices to prove the first inequality, the proof of the second being analogous. Put $v = H_{(1;E)}(u)$. If $v$ does not attain its minimum at any point of $V^{(1)} \setminus V^{(0)}$, then we have $\min v = v(Q) = u(Q) \geq \min u$ for some $Q \in V^{(0)}$, and the first inequality is trivial. Suppose, on the contrary, there exists $Q \in V^{(1)} \setminus V^{(0)}$ at which $v$ attains its minimum. By virtue of (3), since $v(P) \geq v(Q) =: m$ for each point $P$ close to $Q$, we have in fact $v(P) = v(Q)$ for each point $P$ close to $Q$. In particular, $v(P) = m$ for all $P \in V^{(1)} \setminus V^{(0)}$, thus we can repeat the same argument at all $P \in V^{(1)} \setminus V^{(0)}$. But every point of $V^{(0)}$ is close to some point in $V^{(1)} \setminus V^{(0)}$. Thus, $v = m$ on $V^{(1)}$, in particular, since $v = u$ on $V^{(0)}$, $u$ is constant. □

We will now prove as a simple consequence of the maximum principle, that the oscillation of the harmonic extension of $u$ on every 1-cell is estimated by a constant times the oscillation of $u$ on $V^{(0)}$. Let

$$\mathcal{S} := \left\{ u \in \mathbb{R}^{V^{(0)}} : u(P_1) = 0, ||u|| = 1 \right\}. \tag{6}$$

and, given a function $f$ from a nonempty set $A$ to $\mathbb{R}$ let

$$\mathrm{Osc}_A f = \sup_{x,y \in A} |f(x) - f(y)| = \sup_A f - \inf_A f .$$

**Corollary 1.** *There exists* $\gamma \in ]0, 1[$ *such that for all* $u : V^{(0)} \to \mathbb{R}$ *we have* $\sup\limits_{i=1,2,3} \mathrm{Osc}_{V_i}(H_{(1;E)}(u)) \leq \gamma \, \mathrm{Osc}_{V^{(0)}}(u)$.

*Proof.* Suppose $u \in \mathbb{R}^{V^{(0)}}$, $u$ nonconstant. Then, by the strong maximum principle, for every $i = 1, 2, 3$ and for every $P \in V^{(0)}$ we have $\min\limits_{V^{(0)}} u \leq H_{(1;E)}(u)(\psi_i(P)) \leq \max\limits_{V^{(0)}} u$ and the inequalities are strict if $P \neq P_i$. Hence, at least one of the two inequalities is strict for every $P \in V^{(0)}$, and $\mathrm{Osc}_{V_i}(H_{(1;E)}(u)) < \mathrm{Osc}_{V^{(0)}}(u)$ for $i = 1, 2, 3$. Thus, putting

$$\alpha(u) = \frac{\sup\limits_{i=1,2,3} \mathrm{Osc}_{V_i}(H_{(1;E)}(u))}{\mathrm{Osc}_{V^{(0)}}(u)} ,$$

we have $\alpha(u) < 1$. Since $\alpha$ is continuous it has a maximum $\gamma < 1$ on $\mathcal{S}$. Since $\alpha(u + c) = \alpha(u)$ for every $c \in \mathbb{R}$, and $\alpha$ is positively 0-homogeneous, we have

$$\alpha(u) = \alpha\Big(\frac{u - u(P_1)}{||u - u(P_1)||}\Big) \le \gamma$$

for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$. $\square$

Given a function $u : V^{(0)} \to \mathbb{R}$, as hinted above, we want to extend it to a function $v$ on $V^{(\infty)}$ in the following way. Put $v = u$ on $V^{(0)}$, then we define $v$ recursively on $V^{(n+1)} \setminus V^{(n)}$ for $n \ge 0$. For $n = 0$, $v$ is the harmonic extension of $u$ on $V^{(1)}$. More generally, suppose $v$ is already defined on $V^{(n)}$ and extend $v$ on $V^{(n+1)}$. Note that $V^{(n+1)} = \bigcup_{i_1,\dots,i_n=1}^{3} \psi_{i_1,\dots,i_n}(V^{(1)})$. So, we define $v$ separately on every $\psi_{i_1,\dots,i_n}(V^{(1)})$. Let

$$v\big(\psi_{i_1,\dots,i_{n+1}}(P)\big) = H_{(1;E)}\big(v \circ \psi_{i_1,\dots,i_n}\big)\big(\psi_{i_{n+1}}(P)\big) \tag{7}$$

$\forall\, P \in V^{(0)}$, $\forall\, i_1, \dots, i_{n+1} = 1, 2, 3$, in other words, $v \circ \psi_{i_1,\dots,i_{n+1}} = H_{1;E}\big(v \circ \psi_{i_1,\dots,i_n}\big) \circ \psi_{i_{n+1}}$ on $V^{(0)}$. The right-hand side makes sense, as we have already defined $v$ on $V^{(n)}$. In some sense, by identifying a function on $\psi_{i_1,\dots,i_n}(V^{(1)})$ with a function on $V^{(1)}$, (7) defines $v$ on $\psi_{i_1,\dots,i_n}(V^{(1)})$ as the harmonic extension of the restriction of $v$ on $\psi_{i_1,\dots,i_n}(V^{(0)})$. The problem in defining $v$ on $V^{(n+1)}$ by (7), is that we could suspect that a point $Q$ can be represented as a point of $V^{(n+1)}$ in different ways, for example

$$Q = \psi_{i_1,\dots,i_{n+1}}(P) = \psi_{i'_1,\dots,i'_{n+1}}(P'), \tag{8}$$

and that the definition of $v(Q)$ via (7) can depend on such a representation. We are now going to prove that this is not the case, so that (7) actually defines a function $v$. The geometrical reason for this is that we see that different ($n^{th}T$) copies can have as common points only their vertices, so that the points $Q$ in $V^{(n+1)} \setminus V^{(n)}$ can be represented in only one way as in (8). Such an argument can be considered as sufficiently persuasive. However, I will give a formal proof. We have to prove that if (8) holds then

$$H_{(1;E)}\big(v \circ \psi_{i_1,\dots,i_n}\big)\big(\psi_{i_{n+1}}(P)\big) = H_{(1;E)}\big(v \circ \psi_{i'_1,\dots,i'_n}\big)\big(\psi_{i'_{n+1}}(P')\big)\,. \tag{9}$$

If $(i_1, \dots, i_n) = (i'_1, \dots, i'_n)$ then, as every map $\psi_i$ is one-to-one, we must have by (8), $\psi_{i_{n+1}}(P) = \psi_{i'_{n+1}}(P')$, thus (9) holds. Suppose now $(i_1, \dots, i_n) \ne (i'_1, \dots, i'_n)$. Then, it suffices to prove that

$$\psi_{i_{n+1}}(P), \psi_{i'_{n+1}}(P') \in V^{(0)}\,, \tag{10}$$

so that by definition of harmonic extension, we have

$$H_{(1;E)}\big(v \circ \psi_{i_1,\dots,i_n}\big)\big(\psi_{i_{n+1}}(P)\big) = v\Big(\psi_{i_1,\dots,i_n}\big(\psi_{i_{n+1}}(P)\big)\Big)$$

$$= v(Q) = H_{(1;E)}\big(v \circ \psi_{i'_1,\dots,i'_n}\big)\big(\psi_{i'_{n+1}}(P')\big)$$

and (9) holds. In order to prove (10), we need the following lemma.

**Lemma 3.** *Suppose* $(i_1,...,i_n) \neq (i'_1,...,i'_n)$*. Then,*

$$T_{i_1,\dots,i_n} \cap T_{i'_1,\dots,i'_n} \subseteq V_{i_1,\dots,i_n} \cap V_{i'_1,\dots,i'_n}\,.$$

*Proof.* By the definition of $\psi_i$, we have $\psi_i(T) \cap \psi_{i'}(T) \subseteq \psi_i(V^{(0)}) \cap \psi_{i'}(V^{(0)})$ if $i, i' = 1,2,3,\ i \neq i'$. Thus, if $i_m \neq i'_m$, and $i_l = i'_l$ for all $l < m$, in view of the trivial fact that every $\psi_i$ maps $T$ into itself, we have

$$\psi_{i_m,\dots,i_n}(T) \cap \psi_{i'_m,\dots,i'_n}(T) \subseteq \psi_{i_m}(T) \cap \psi_{i'_m}(T) \subseteq \psi_{i_m}(V^{(0)}) \cap \psi_{i'_m}(V^{(0)}).$$

Clearly, every $\psi_i$ maps $T \setminus V^{(0)}$ into itself, hence so does $\psi_{i_{m+1},\dots,i_n}$. It follows that

$$\psi_{i_{m+1},\dots,i_n}(T) \cap V^{(0)} \subseteq \psi_{i_{m+1},\dots,i_n}(V^{(0)})$$

hence, using also the fact that every $\psi_i$ is one-to-one,

$$\begin{aligned}\psi_{i_m,\dots,i_n}(T) \cap \psi_{i'_m,\dots,i'_n}(T) &\subseteq \psi_{i_m,\dots,i_n}(T) \cap \psi_{i_m}(V^{(0)})\\ &= \psi_{i_m}\Big(\psi_{i_{m+1},\dots,i_n}(T) \cap V^{(0)}\Big) \subseteq \psi_{i_m,\dots,i_n}(V^{(0)})\,,\end{aligned}$$

$$\begin{aligned}\psi_{i_1,\dots,i_n}(T) \cap \psi_{i'_1,\dots,i'_n}(T) = \psi_{i_1,\dots,i_{m-1}}\big(\psi_{i_m,\dots,i_n}(T)\big) \cap \psi_{i_1,\dots,i_{m-1}}\big(\psi_{i'_m,\dots,i'_n}(T)\big)\\ \subseteq \psi_{i_1,\dots,i_n}(V^{(0)})\,.\end{aligned}$$

The same argument is valid for $i'_1,...,i'_n$ in place of $i_1,...,i_n$, thus we have proved the lemma. □

Now, since we have supposed $(i_1,...,i_n) \neq (i'_1,...,i'_n)$, and the maps $\psi_i$ are one-to-one, in view of (8) and Lemma 3, (10) follows at once, and thus (7) defines a function on $V^{(\infty)}$. We now want to prove that $v$ is uniformly continuous on $V^{(\infty)}$, and hence it can be extended continuously on $K$. We will get this by proving that the oscillation of $v$ on the $n$-cells tends to 0 as $n$ tends to $\infty$. Let us give the following definition. For $f : V^{(\infty)} \to \mathbb{R}$, define $\mathrm{Osc}_{,n} f = \sup_{i_1,..,i_n=1,2,3} \mathrm{Osc}_{V^{(\infty)}_{i_1,\dots,i_n}} f$. We have

**Lemma 4.** $\mathrm{Osc}_{,n}(v) \underset{n\to+\infty}{\longrightarrow} 0$.

*Proof.* By (7) and Corollary 1, we have

$$\mathrm{Osc}_{V_{i_1,\dots,i_{n+1}}}(v) \le \gamma \mathrm{Osc}_{V_{i_1,\dots,i_n}}(v)$$

for all $n \in \mathbb{N}$, $i_1, ..., i_{n+1} = 1, 2, 3$, thus, for all $i_1, ..., i_n = 1, 2, 3$ we have

$$\mathrm{Osc}_{V_{i_1,\dots,i_n}}(v) \le \gamma^n \mathrm{Osc}_{V^{(0)}}(u)\,.$$

Now, if $P \in V^{(\infty)}_{i_1,\dots,i_n}$, we have $P = \psi_{i_1,\dots,i_n,i_{n+1},\dots,i_h}(Q)$ for some $Q \in V^{(0)}$ and $i_{n+1}, ..., i_h = 1, 2, 3$. Thus, using (7) again and the maximum principle, a recursive argument yields

$$\min_{V_{i_1,\dots,i_n}} v \le\ v(P) \le \max_{V_{i_1,\dots,i_n}} v$$

and thus

$$\mathrm{Osc}_{V^{(\infty)}_{i_1,\dots,i_n}}(v) \le \mathrm{Osc}_{V_{i_1,\dots,i_n}}(v) \le \gamma^n \mathrm{Osc}_{V^{(0)}}(u)\,.$$

Since this holds for every $i_1, ..., i_n$, we have proved the lemma. □

**Corollary 2.** *$v$ is uniformly continuous on $V^{(\infty)}$.*

*Proof.* Let $\varepsilon > 0$ be given. Let $n$ be such that $\mathrm{Osc}_{,n}(v) < \frac{\varepsilon}{2}$. Since

$$V \subseteq V^{(\infty)} \subseteq T\,,$$

using Lemma 3 we get that for every $i_1, ..., i_n$, $i'_1, ..., i'_n = 1, 2, 3$, either $V^{(\infty)}_{i_1,\dots,i_n}$ and $V^{(\infty)}_{i'_1,\dots,i'_n}$ have nonempty intersection, or $T_{i_1,\dots,i_n}$ and $T_{i'_1,\dots,i'_n}$ are disjoint, and so, they being compact, they have a positive minimum distance. Let $\delta > 0$ be less than the minimum of the distance of disjoint $T_{i_1,\dots,i_n}$ and $T_{i'_1,\dots,i'_n}$. If $P, P' \in V^{(\infty)}$ and $d(P, P') < \delta$, we have $P \in V^{(\infty)}_{i_1,\dots,i_n}$, $P' \in V^{(\infty)}_{i'_1,\dots,i'_n}$ for some $i_1, ..., i_n$, $i'_1, ..., i'_n$ and there exists $Q \in V^{(\infty)}_{i_1,\dots,i_n} \cap V^{(\infty)}_{i'_1,\dots,i'_n}$. Hence, $|v(P) - v(Q)| < \frac{\varepsilon}{2}$ and $|v(P') - v(Q)| < \frac{\varepsilon}{2}$, thus $|v(P) - v(P')| < \varepsilon$. □

We will call the continuous extension on $K$ of $v$ defined by (7), the *harmonic extension* of $u$ on $K$, and we will denote it by $H_{(\infty;E)}(u)$. I explicitly state the following lemma which is implicit in the proof of Lemma 4.

**Lemma 5.** *For every $u : V^{(0)} \to \mathbb{R}$ we have*

$$\min u \le H_{(\infty;E)}(u) \le \max u\,.$$

*Proof.* Put $v = H_{(\infty;E)}(u)$. By (7) and the maximum principle we have

$$\min_{V_{i_1,\dots,i_n}} v \le \min_{V_{i_1,\dots,i_{n+1}}} v \le \max_{V_{i_1,\dots,i_{n+1}}} v \le \max_{V_{i_1,\dots,i_n}} v$$

so that, by a recursive argument we get

$$\min_{V^{(0)}} u = \min_{V^{(0)}} v \le \min_{V_{i_1,\dots,i_n}} v \le \max_{V_{i_1,\dots,i_n}} v \le \max_{V^{(0)}} v = \max_{V^{(0)}} u \,,$$

hence, $\min u \le v(P) \le \max u$, for every $P \in V^{(\infty)}$, thus by continuity for every $P \in K$. □

The use of the harmonic extension is illustrated in the following theorem.

**Theorem 2.** *For every $u \in \mathbb{R}^{V^{(0)}}$, we have*

$$E^{\Sigma}_{(\infty)}(H_{(\infty;E)}(u)) = E(u) \,.$$

*Proof.* Put $H_{(\infty;E)}(u) =: v$. Using also (5), we get

$$E^{\Sigma}_{(n+1)}(v) = \frac{1}{\rho^{n+1}} \sum_{i_1,\dots,i_{n+1}=1}^{3} E(v \circ \psi_{i_1,\dots,i_{n+1}})$$

$$= \frac{1}{\rho^{n+1}} \sum_{i_1,\dots,i_n=1}^{3} \Big( \sum_{i_{n+1}=1}^{3} E\big(H_{(1;E)}(v \circ \psi_{i_1,\dots,i_n}) \circ \psi_{i_{n+1}}\big)\Big)$$

$$= \frac{1}{\rho^{n}} \sum_{i_1,\dots,i_n=1}^{3} E^{\Sigma}_{(1)}\big(H_{(1;E)}(v \circ \psi_{i_1,\dots,i_n})\big)$$

$$= \frac{1}{\rho^{n}} \sum_{i_1,\dots,i_n=1}^{3} E(v \circ \psi_{i_1,\dots,i_n}) = E^{\Sigma}_{(n)}(v) \,,$$

for all $n \in \mathbb{N}$, thus $E^{\Sigma}_{(n)}(H_{(\infty;E)}(u)) = E(u)$ for all $n \in \mathbb{N}$. □

Note that by Theorem 2 we see that $H_{(\infty;E)}(u)$ minimizes $E^{\Sigma}_{(\infty)}$ among the functions defined on $K$ which amount to $u$ on $V^{(0)}$. We need the following simple lemma.

**Lemma 6.** *We have*

$$K = \bigcup_{i_1,\dots,i_n=1}^{3} \psi_{i_1,\dots,i_n}(K) \,.$$

*Proof.* We have $V^{(\infty)} = \bigcup_{i_1,\dots,i_n=1}^{3} \psi_{i_1,\dots,i_n}(V^{(\infty)})$. It now suffices to observe that $K = \overline{V^{(\infty)}}$ and that $\psi_{i_1,\dots,i_n}$ are continuous. □

Given a continuous function $v$ on $K$, we can now construct for every $m \in \mathbb{N}$, a function denoted by $v_{(m;E)}$, defined on $K$, which is in some sense the harmonic extension of the restriction of $v$ on $V^{(m)}$. Namely, for every $P \in K$, put

$$v_{(m;E)}\big(\psi_{i_1,\dots,i_m}(P)\big) = H_{(\infty;E)}(v \circ \psi_{i_1,\dots,i_m})(P)\,.$$

Thanks to Lemma 3, such a definition is correct, and the function $v_{(m;E)}$ is continuous, its restriction on every $K_{i_1,\dots,i_m}$ being continuous and the sets $K_{i_1,\dots,i_m}$ being closed subsets of $K$ covering $K$. Moreover, using an argument like that in Theorem 2, we have

$$E^{\Sigma}_{(\infty)}(v_{(m;E)}) = E^{\Sigma}_{(m)}(v) < +\infty,$$

so that $v_{(m;E)}$ has finite energy. We are going to prove that $v_{(n;E)} \underset{n\to\infty}{\longrightarrow} v$ uniformly. In fact, for every $Q \in K$ let $i_1, ..., i_n = 1, 2, 3,\ P \in K$ be such that $Q = \psi_{i_1,\dots,i_n}(P)$. By the definition of $v_{(n;E)}$ and Lemma 5, we have $v_{(n;E)}(Q), v(Q) \in [\inf\limits_{K_{i_1,\dots,i_n}} v, \sup\limits_{K_{i_1,\dots,i_n}} v]$. Since

$$\text{diam} K_{i_1,\dots,i_n} \le (\frac{1}{2})^n \text{diam} K\,, \tag{11}$$

(we have in fact the equality in (11)) and $v$ is uniformly continuous, it follows that $v_{(n;E)} \underset{n\to\infty}{\longrightarrow} v$ uniformly, as claimed. In conclusion,

**Theorem 3.** *The set of functions with finite energy is dense in $C(K)$.* □

## 3 Dirichlet forms on finitely ramified fractals

In this section we extend the construction described in the previous section from the case of the Gasket to the case of more general fractals. First, I describe other examples of fractals. The *Vicsek set* can be constructed analogously to the Gasket, by putting $K_0$ to be a square, $P_1, P_2, P_3, P_4$ its vertices, and $P_5$ its centre, and $\psi_i(x) = P_i + \frac{1}{3}(x - P_i)$, and $K_{n+1} = \bigcup\limits_{i=1}^{5} \psi_i(K_n)$, then using (1). The *Sierpinski Carpet* is also constructed starting from a square $K_0$, then splitting it into nine small squares of edge $\frac{1}{3}$, and considering the eight similarities carrying it into the small squares but the central one. The (Lindstrøm) *Snowflake* is obtained from a hexagon, and seven similarities, six of them having for fixed points the vertices of the hexagon, and the other one the centre of the hexagon. The *tree-like Gasket* is similar to the Gasket, but two of the three small triangles are disjoint. I finally also recall the celebrated *Cantor set*, where the initial set $K_0$ is a segment-line, and there are two similarities having the end points of $K_0$ for fixed points and of factor $\frac{1}{3}$. What could be a general definition of fractal? In all previous cases the set in $\mathbb{R}^\nu$ is obtained using an initial set and finitely many contractive (i.e, having a factor $< 1$) similarities. By a slightly deeper investigation we realize that the initial set is not essential in this construction, as, for example in the case of the Gasket, we can start with $V^{(0)}$ instead of with $T$. In Figures 2 and 3, we describe the previous fractals, by depicting $K_1$.

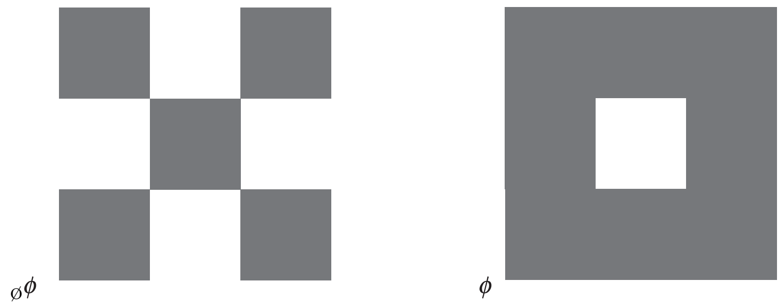


**Fig. 2.** The Vicsek set and the Sierpinski Carpet

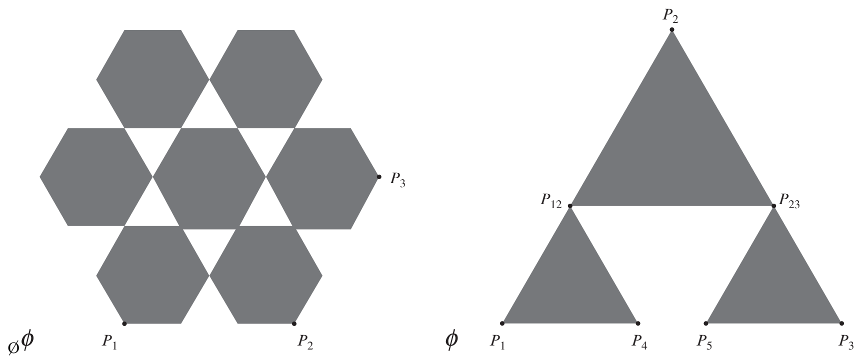


**Fig. 3.** The Snowflake and the tree-like Gasket

What is the relationship between the similarities and the fractal obtained? The answer is: if $\psi_1, ..., \psi_k$ are the similarities, then $K$ is the unique nonempty compact subset of $\mathbb{R}^\nu$ such that $K = \bigcup_{i=1}^{k} \psi_i(K)$. In order to realize such a construction, we need some preliminary considerations. Fix $\nu = 1, 2, 3, ...$ in the following, and let $\mathcal{K}$ be the set of nonempty compact subsets of $\mathbb{R}^\nu$. We equip $\mathcal{K}$ with the so-called Hausdorff distance, namely $d_H(C_1, C_2) = \sup\{d(x, C_2) : x \in C_1, d(y, C_1) : y \in C_2\}$. Such a distance in some sense measures how much close are two sets. It is easy to prove that it is in fact a metric on $\mathcal{K}$. We now prove that it is complete.

**Theorem 4.** $(\mathcal{K}, d_H)$ *is complete.*

*Proof.* Suppose $C_n$ is a Cauchy sequence in $\mathcal{K}$, and prove that it has a limit. There exists a subsequence of $C_n$, that we will call $D_n$ such that $d_H(D_n, D_{n+1}) < \frac{1}{2^n}$ for all $n$. We will prove that $D_n$ has a limit and this suffices to conclude the proof. Let

$$D = \left\{x \in \mathbb{R}^\nu : \exists\, x_n \in D_n \text{ with } x = \lim_{n\to\infty} x_n\right\}.$$

We will prove that $D$ is nonempty and compact and that $D = \lim_{n\to\infty} D_n$. In order to achieve this, observe that for all $n = 1, 2, 3, ...$ we have

$$\forall\, x \in D_n\ \exists\, y \in D : d(x, y) \le \frac{1}{2^{n-1}}, \tag{12}$$

$$\forall\, n\ \forall\, x \in D\ \exists\, x_n \in D_n : d(x, x_n) \le \frac{3}{2^n}. \tag{13}$$

In order to prove (12), note that by the definition of $d_H$, there exists a sequence $(x_1, ..., x_m, ...)$ so that $x = x_n$, and for all $m = 1, 2, 3, ...,$ $x_m \in D_m$ and $d(x_m, x_{m+1}) < \frac{1}{2^m}$. Then $x_m$ is a Cauchy sequence and its limit $y$ belongs to $D$. Clearly, by the triangular inequality, $d(x_n, x_m) < \frac{1}{2^{n-1}}$ for all $m \ge n$, hence $d(x, y) \le \frac{1}{2^{n-1}}$.

Let us now prove (13). Let $x \in D$, $y_m \in D_m$ with $x = \lim_{m\to\infty} y_m$, and let $m > n$ be such that $d(x, y_m) \le \frac{1}{2^n}$. As above we find $x_n \in D_n$ such that $d(x_n, y_m) < \frac{1}{2^{n-1}}$, and (13) follows at once. From (12) we see, in particular, that $D$ is nonempty. Since $D_n$ are bounded, in view of (13), so is $D$. In order to prove that $D$ is closed, suppose $x_n \in D$ and $x = \lim_{n\to\infty} x_n$, and prove that $x \in D$. Using (13) we find $y_n \in D_n : d(y_n, x_n) \le \frac{3}{2^n}$. Hence, $x = \lim_{n\to\infty} y_n$, and $x \in D$. From (12) and (13) it immediately follows that $D = \lim_{n\to\infty} D_n$. □

Suppose now $\psi_i$ are contractive similarities in $\mathbb{R}^\nu$, for $i = 1, ..., k$, with factors $r_i \in ]0, 1[$, i.e. we have

$$||\psi_i(x) - \psi_i(y)|| = r_i||x - y|| \quad \forall\, x, y \in \mathbb{R}^\nu.$$

Let $\Phi : \mathcal{K} \to \mathcal{K}$ be defined as

$$\Phi(C) = \bigcup_{i=1}^k \psi_i(C).$$

We are searching for a fixed point of $\Phi$. We have the following

**Theorem 5.** *$\Phi$ is a contraction of $(\mathcal{K}, d_H)$, hence it has a unique fixed point $K$, and moreover, $\Phi^n(A) \underset{n\to\infty}{\longrightarrow} K$ for every $A \in \mathcal{K}$.*

*Proof.* Let $r = \max r_i$. Then, clearly, for every $A, B \in \mathcal{K}$ and for every $i = 1, ..., k$ we have $d_H(\psi_i(A), \psi_i(B)) \le r\ d_H(A, B)$. From the definition of $d_H$ it easily follows that $\Phi$ is a contraction with factor $r$. □

In some important particular cases, we can also give a better characterization of $\Phi^n(A)$. For example, if $T \in \mathcal{K}$ is such that $\Phi(T) \subseteq T$, we easily see by recursion that $\Phi^n(T)$ is a decreasing sequence of sets. It follows that $\bigcap_{n=0}^{\infty} \Phi^n(T) = K$. In the case of the Gasket if $T$ is the triangle, we see that $\Phi^n(T) = K_n$ defined as in (1), thus we find again $K = \bigcap_{n=0}^{\infty} T_n$. On the other hand, if we denote by $F$ the set of the fixed points of $\psi_i$ if $\emptyset \neq V \subseteq F$, we see that $V \subseteq \Phi(V)$, thus $\Phi^n(V) =: V^{(n)}$ is an increasing sequence, and we easily get $\bigcup_{n=0}^{\infty} V^{(n)} = K$. I stated this in the case of the Gasket. However, in general, we do not require that $V$ is the set of the fixed points of *all* $\psi_i$ but of some $\psi_i$. This remark is important for the following. We will say that the set $K$ as in Theorem 5 is the fractal (or self-similar set) generated by the set $\Psi = \{\psi_i : i = 1, ..., k\}$ of contractive similarities. Note that a fractal can be generated by different sets of similarities. However, for the following we fix a set $K$ as above with the associated similarities $\psi_1, ..., \psi_k$, and call $P_i$ the fixed point of $\psi_i$ for $i = 1, ..., k$, and put $F = \{P_i : i = 1, ..., k\}$. We have so explained what we mean by fractal. The next problem is how to construct an energy on it. We will do that by imitating the construction on the Gasket. However, as we will see, that kind of construction is not possible on every fractal, but we have in fact to require some additional hypotheses. Before analyzing what properties we need, we are going to give a more precise notion of the so far vague word *energy*. We require of course that an energy is a functional defined on $C(K)$ which is nonnegative, quadratic, and which takes the value 0 on the constant functions. It also appears to be natural to assume that it takes the value 0 only on constant functions and that it is finite on a dense set of continuous functions. Moreover, it seems to be natural to require a property of compatibility with the fractal structure. The need of giving a notion of energy generalizing the Dirichlet integral has lead to the notion of *Dirichlet form*. Usually, a Dirichlet form is defined on an $L^2$ space. Here, in order to avoid problems related to the construction of measures on the fractal, I prefer to define it on $C(K)$, although this is, in some sense, less natural.

**Definition 1.** *We say that a functional $\mathcal{E}$ is a* good Dirichlet form *if it satisfies the following properties*

*a) $\mathcal{E}$ is a quadratic form from $C(K)$ to $[0, +\infty]$ in the sense that there exists a linear subspace $Z$ of $C(K)$ such that $\mathcal{E}(v) < +\infty$ if and only if $v \in Z$, and there exists $\hat{\mathcal{E}} : Z \times Z \to \mathbb{R}$ bilinear and symmetric such that*

$$\mathcal{E}(v) = \hat{\mathcal{E}}(v, v) \geq 0$$

*for all $v \in Z$.*

*b) $\mathcal{E}((v \wedge 1) \vee 0) \leq \mathcal{E}(v) \quad \forall\, v \in C(K)$ (Markov property).*

*c) $\mathcal{E}$ is lower semicontinuous (with respect to the $L^\infty$ topology).*

*d)* $\mathcal{E}(v+c) = \mathcal{E}(v) \quad \forall\, v \in C(K), \forall\, c \in \mathbb{R}$, *(thus, $\mathcal{E}$ is* $0$ *on all constant functions).*

*e) $\mathcal{E}$ is irreducible, i.e., $\mathcal{E}(v) = 0 \Rightarrow v$ constant.*

*f) There exists a set $\mathcal{H} \subseteq C(K)$ such that $\mathcal{H}$ is dense in $C(K)$ with respect to the $L^\infty$ topology, and $\mathcal{E}(v) < +\infty$ for every $v \in \mathcal{H}$.*

*g)* $\exists\, \rho > 0 : \mathcal{E}(v) = \frac{1}{\rho} \sum_{i=1}^{k} \mathcal{E}(v \circ \psi_i)$

Properties a), b), c) characterize the Dirichlet forms, d), e), f) are in some sense properties of regularity of a Dirichlet form, and g) is the self-similarity property. The functional $E^{\Sigma}_{(\infty)}$ constructed on the Gasket in Section 2 is a good Dirichlet form. Indeed, a) is trivial, b) easily follows from the analogous property of the function $(x, y) \mapsto (x-y)^2$ (i.e. if $s(t) = (t \wedge 1) \vee 0$, then $(s(x)-s(y))^2 \le (x-y)^2$); c) follows from the fact that $E^{\Sigma}_{(\infty)}$ is the sup of the continuous functionals $E^{\Sigma}_{(n)}$, d) is trivial, e) and f) have been proved in Section 2, and g) can be easily proved with $\rho = \frac{3}{5}$ on the base of the definition of $E^{\Sigma}_{(n)}$. We are now trying to imitate the construction of $E^{\Sigma}_{(\infty)}$ on a general fractal. To do this, we have to restrict the class of fractals considered, requiring that analogous properties to those used in the construction on the Gasket hold in our fractals. Here, in order to simplify the presentation, we will not try to define the widest class of fractals suitable for these considerations, but we will restrict ourselves to consider a class of fractals which is at the same time simpler to define and sufficient to include the most usual cases. In the case of Gasket, we used the fact that $\psi_i(T \setminus V) \subseteq T \setminus V$. Such a property was crucial in the construction of the harmonic extension on $K$ of a function defined in $V$, which in turn allowed us to prove property f) of Def. 1. Also, we implicitly used the fact that the points $P_j$ are different from each other, and that the points $\psi_i(P_j)$ are not in $V^{(0)}$ unless $i = j$. So, we are lead to require the following property which includes the previous two:

**Definition 2.** *We say that $K$ has the (strong)* nesting property *if $P_{j_1} \neq P_{j_2}$ when $j_1 \neq j_2$, and there exists a set $V = V^{(0)} \subseteq F$, $V = \{P_1, ..., P_N\}$, $2 \le N \le k$, so that, putting $T = \mathrm{co}V$, we have $\psi_i(T) \subseteq T$ and $\psi_i(x) \notin V^{(0)}$ if $x \in T \setminus \{P_i\}$, and in addition*

$$\psi_i(T) \cap \psi_{i'}(T) = \psi_i(V) \cap \psi_{i'}(V) \text{ if } i, i' = 1, ..., k, i \neq i' \,. \tag{14}$$

Note that we used (14) in Lemma 3. In the case of the Gasket, $V$ and $T$ are as in Section 2. In the absence of the nesting property, we cannot even conclude that $E^{\Sigma}_{(\infty)}$ is finite for some nonconstant function. It follows that $\psi_i(T \setminus V^{(0)}) \subseteq T \setminus V^{(0)}$, for all $i = 1, ..., k$. Note that the nesting property implies in particular that every $n$-cell contains at most one point of $V^{(0)}$ for $n > 0$. In addition,

$$P_h = \psi_i(P_j), \quad h, j = 1, ..., N,\ i = 1, ..., k \Rightarrow i = j = h\,.$$

This means that every $P_h \in V^{(0)}$ belongs to precisely one 1-cell, i.e., $V_h(:= \psi_h(V^{(0)}))$. The phrase *strong nesting property* is more appropriate, as usually nesting property has a weaker meaning, and properly is a weaker version of (14), but in the following we conventionally omit the word strongly. In some sense, in Def. 2, (14) is the most characterizing property, in which it distinguishes the most usual fractals. In fact, the Vicsek set, the tree-like Gasket and the Snowflake have the nesting property, but the Carpet has not, as (14) does not hold. In our examples we have $N = k = 3$ in the Gasket and in the tree-like Gasket, $N = 4$, $k = 5$ in the Vicsek set, $N = 6$, $k = 7$ in the Snowflake, so that we can in fact have $N < k$. Another property we have to require is suggested by the Cantor set. There, if we try to imitate the definition of $M_1(E)$ we obtain 0, as, for every function defined on $V$, which in this case is the set of the end-points of the segment-line, it can be extended on $V^{(1)}$ to a function which is constant on each 1-cell. The reason is that the Cantor set is too much disconnected, not so in a topological sense but in the combinatorial sense that the 1-cells are disjoint. In order to give a precise notion of connectedness, we recall the following definitions about graphs.

*A graph is a pair $(V, W)$ where $V$ is a nonempty set and $W$ is a subset of the set of the subsets of $V$ having precisely two elements. The elements of $W$ will be called the edges of the graph. We will say that $P, Q \in V$ are close (in $(V, W)$) if $\{P, Q\}$ is an edge. We will say that $P, Q \in V$ are connected (in $(V, W)$) if there exist $n = 1, 2..., P_1, ..., P_n \in V$ such that $P_1 = P$, $P_n = Q$, and $\{P_i, P_{i+1}\} \in W$ for $i = 1, ..., n-1$. In such a case, we will say that $(P_1, ..., P_n)$ is a path that connects $P$ and $Q$ (in $(V, W)$) and has length $n$. We will say that $(V, W)$ is connected, if any two points in $V$ are connected. When $V$ is clear from the context we can identify the graph with $W$, and say for example that $W$ is connected.*

Now define $A_{i_1,...,i_n} = \psi_{i_1,...,i_n}(A)$ for $A \subseteq K$ and $i_1, ..., i_n = 1, ..., k$, where $\psi_{i_1,...,i_n}$ is an abbreviation for $\psi_{i_1} \circ ... \circ \psi_{i_n}$, and put $V^{(n)} = \bigcup_{i_1,..,i_n=1}^{k} V_{i_1,...,i_n}$, $V^{(\infty)} = \bigcup_{n=0}^{\infty} V^{(n)}$ as in the case of Gasket. We explicitly note that

$$V^{(0)} \subseteq V^{(n)} \subseteq V^{(\infty)} \subseteq K \subseteq T,$$

and $V^{(n)}$ is increasing with respect to $n$. Then we say that our fractal is connected if the following graph is connected: $\mathcal{G}_1 = (V^{(1)}, W)$ where $W$ is the set of $\{\psi_i(P_{j_1}), \psi_i(P_{j_2})\}$ with $i = 1, ..., k$, $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$. In other words, we require that for every $P, Q \in V^{(1)}$ there exists a finite sequence $P_1, ..., P_m$ of points in $V^{(1)}$ such that, $P = P_1$, $Q = P_m$, and for every $r = 1, ..., m-1$, $P_r, P_{r+1}$ belong to some 1-cell, or also, that any two points in $V^{(1)}$ can be connected by a path in $V^{(1)}$ whose edges are contained in some 1-cell (depending on the edge). It can be easily verified that the Vicsek set, the tree-like Gasket and the Snowflake are connected. We will say that a fractal is (strongly) finitely ramified if both it has the nesting property and it is connected. As for the nesting property, we will omit the word strongly for sake of simplicity, although, the usual definition of finitely ramified fractals is more general.

We will construct a good Dirichlet form on our fractal, which from now on will be always assumed to be finitely ramified. At first glance, in the construction in Section 2 we used other properties of the Gasket. For example, in the proof of the maximum principle, we used the fact, that every point in $V^{(1)} \setminus V^{(0)}$ is close to a point close to a fixed point in $V^{(0)}$. However, this tells in other words that a point in $V^{(1)} \setminus V^{(0)}$ and a point in $V^{(0)}$ are connected in $\mathcal{G}_1$ by a path of length $\leq 3$, and the connectedness can well replace this assumption. A more serious difficulty is that, when we proved that $M_1(E)$ is a multiple of $E$ we heavily used the very strong symmetry of the Gasket, for example in the Snowflake there is no reason that the coefficients of $(u(P_1) - u(P_2))^2$ and of $(u(P_1) - u(P_3))^2$ are the same, as $P_1$ and $P_2$ are connected in $V^{(1)}$ in an essentially different way from $P_1$ and $P_3$. In order to avoid such a problem, we will consider more general quadratic forms on $V^{(0)}$. Suppose $K$ is a finitely ramified fractal with similarities $\psi_1, ..., \psi_k$. We define $\mathcal{D}$ to be the set of functionals from $\mathbb{R}^{V^{(0)}}$ into $\mathbb{R}$ satisfying the following property:

*there exist* $c_{j_1,j_2}(E)(= c_{j_1,j_2}) \geq 0$ *(*$j_1 \neq j_2$*) with* $c_{j_1,j_2} = c_{j_2,j_1}$ *such that*

$$E(u) = \sum_{1 \leq j_1 < j_2 \leq N} c_{j_1,j_2}(u(P_{j_1}) - u(P_{j_2}))^2 \tag{15}$$

*for all* $u : V^{(0)} \to \mathbb{R}$.

Moreover, we define $\widetilde{\mathcal{D}}$ to be the set of those $E \in \mathcal{D}$ which are irreducible, i.e., $E(u) = 0$ if and only if $u$ is constant. Concerning the previous definitions, we are only interested in $\widetilde{\mathcal{D}}$. However, in some cases, we will also need to consider the set $\mathcal{D}$ which has for example the advantage that it is in some sense a closed set. The difference with respect to the case of the Gasket is that in this case we consider forms with possibly different coefficients. Now, we define $S_n(E)$ and $M_n(E)$ as in Section 2, with the obvious variant that the indices in the sum in the definition of $S_n(E)$ vary from 1 to $k$ instead of from 1 to 3. In order to imitate the construction in Section 2, we need an $E \in \widetilde{\mathcal{D}}$ such that there exists $\rho > 0$ with $M_1(E) = \rho E$, in other words, we have to prove the existence of an eigenvector for the operator $M_1 : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$, which as we will see, is in general, nonlinear. Before discussing this problem, however, we need a more detailed analysis of the previous notions. For example, I have not proved that $M_1$ maps in fact $\widetilde{\mathcal{D}}$ into $\widetilde{\mathcal{D}}$. First of all, we note that the coefficients of $E$ are unique (this enables us to use the notation $c_{j_1,j_2}(E)$). This is a consequence of the following remark.

*Remark 1.* Given $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$, let $u_{j_1,j_2}$, $v_{j_1,j_2}$ be the functions in $\mathbb{R}^{V^{(0)}}$ that take the value 0 at every $P$ different from $P_{j_1}, P_{j_2}$, and such that

$$u_{j_1,j_2}(P_{j_1}) = u_{j_1,j_2}(P_{j_2}) = v_{j_1,j_2}(P_{j_1}) = 1, \quad v_{j_1,j_2}(P_{j_2}) = -1\,.$$

Then, if $E$ is defined as in (15) (*not necessarily* $c_{j_1,j_2} \geq 0$), we must have $c_{j_1,j_2} = \frac{1}{4}\big(E(v_{j_1,j_2}) - E(u_{j_1,j_2})\big)$, as can be easily verified.

We now want to prove that $M_1$ maps in fact $\widetilde{\mathcal{D}}$ into $\widetilde{\mathcal{D}}$. In the case of Gasket, we evaluated precisely $M_1(E)$. We did this by solving explicitly the system in (3). Clearly, this is not possible in the general case. However, it is not necessary to solve explicitly such a kind of system, but it is sufficient to prove that it has a unique solution. For in such a case it depends linearly on $u$ and we can proceed as in Section 2. We need some further considerations on graphs in $V^{(1)}$.

*Remark 2.* It is easy to see that the irreducibility condition for $E \in \mathcal{D}$ amounts to the fact that the graph on $V^{(0)}$, whose edges are the sets $\{P_{j_1}, P_{j_2}\}$ such that $c_{j_1,j_2} > 0$, is connected. We will denote such a graph by $\mathcal{G}(E)$. Note that, roughly speaking, the irreducibility condition amounts to the fact that there are not too many coefficients equal to 0. For example, if $N = 3$, this means that at most one of the coefficients $c_{1,2}, c_{1,3}, c_{2,3}$ is 0.

Fix $E \in \widetilde{\mathcal{D}}$. We call $\mathcal{G}_1(E)$ the graph in $V^{(1)}$ whose edges are the sets of the form $\{\psi_i(P_{j_1}), \psi_i(P_{j_2})\}$ with $i = 1, ..., k,\ j_1, j_2 = 1, ..., N,\ j_1 \neq j_2$ and $c_{j_1,j_2} > 0$. We say that two points $Q$ and $Q'$ in $V^{(1)}$ close (resp. $E$-close) if they are close in $\mathcal{G}_1$ (resp. in $\mathcal{G}_1(E)$). So, two distinct points are close if they lie in the same cell. We say that $Q$ and $Q'$ are connected (resp. $E$-connected) if they are connected in $\mathcal{G}_1$ (resp. in $\mathcal{G}_1(E)$), i.e., if there exists a path $(Q_1, ..., Q_m)$ with $Q_1, ..., Q_m \in V^{(1)}$, $m \geq 1$, and $Q_1 = Q$, $Q_m = Q'$, and $Q_i$ and $Q_{i+1}$ close (resp. $E$-close). In such a case we say that the path connects (resp. $E$-connects) $Q$ to $Q'$. We say that $Q$ and $Q'$ are strongly connected (resp. strongly $E$-connected), and that the path strongly connects (resp. strongly $E$-connects) $Q$ to $Q'$, if we can also assume $Q_2, ..., Q_{m-1} \notin V^{(0)}$. Note however, that by our assumptions any two points in $V^{(1)}$ are connected.

*Remark 3.* It easily follows from Remark 2 that any two points that lie in the same 1-cell $V_i$ are $E$-connected by a path $(Q_1, ..., Q_m)$ with $Q_h \in V_i$ for each $h$. It follows that, if $i > N$, so that $V_i$ contains no points of $V^{(0)}$, then any two points in $V_i$ are strongly $E$-connected. If, instead, $i \leq N$, so that $P_i$ is the unique point in $V^{(0)} \cap V_i$, then any point in $V_i$ is strongly $E$-connected to $P_i$.

**Lemma 7.** *Every point $Q \in V^{(1)}$ is strongly $E$-connected with some point of $V^{(0)}$.*

*Proof.* Fix $Q' \in V^{(0)}$. Since the fractal is connected, there exist $Q_1, ..., Q_m \in V^{(1)}$ such that $Q_1 = Q$, $Q_m = Q'$ and for every $h = 1, ..., m-1$, $Q_h$ and $Q_{h+1}$ are close. Thus, by Remark 3, $Q_h$ and $Q_{h+1}$ are $E$-connected, and therefore $Q$ and $Q'$ are $E$-connected. In a path $(\tilde{Q}_1, ..., \tilde{Q}_l)$ $E$-connecting $Q$ and $Q'$, let $Q''$ be the first element $\tilde{Q}_h \in V^{(0)}$. Then $Q$ and $Q''$ are strongly $E$-connected. □

Note that, unlike the case of the Gasket, in general we cannot conclude that $Q$ is strongly $E$-connected with *any* point of $V^{(0)}$. For example in the tree-like Gasket, if $c_{j_1,j_3} = 0$ the point $P_1$ is not strongly $E$-connected with $P_3$. As in Section 2, we define $\mathcal{L}(n,u) = \{v \in \mathbb{R}^{V^{(n)}} : v = u \text{ on } V^{(0)}\}$. We can characterize the minimum point of $M_1(E)$ in $\mathcal{L}(1,u)$ like in Section 2.

**Lemma 8.** *If $u \in \mathbb{R}^{V^{(0)}}$, then a function $v \in \mathcal{L}(1,u)$ satisfies $M_1(E)(u) = S_1(E)(v)$ if and only if*

$$\sum c_{j_1,j_2}\Big(v(P) - v\big(\psi_i(P_{j_2})\big)\Big) = 0, \quad \forall P \in V^{(1)} \setminus V^{(0)} \tag{16}$$

*where the sum is extended over all $i = 1,...,k$, $j_1, j_2 = 1,...,N$ such that $j_1 \neq j_2$ and $P = \psi_i(P_{j_1})$.* $\square$

**Lemma 9.** *Suppose $u \in \mathbb{R}^{V^{(0)}}$ and $v \in \mathcal{L}(1,u)$ satisfies $M_1(E)(u) = S_1(E)(v)$. Suppose $P \in V^{(1)} \setminus V^{(0)}$ and $v(P) = \max v$ or $v(P) = \min v$. Then, we have $v(Q) = v(P)$ whenever $Q \in V^{(1)}$ is strongly $E$-connected to $P$.*

*Proof.* Suppose for example $v(P) = \max v =: M$. It follows from the hypothesis that $v(P') = v(P)$ if $P'$ is $E$-close to $P$, as, in the contrary case, the left-hand side in (16) would be strictly positive. Now, let $(Q_1, ..., Q_m)$ be a path strongly $E$-connecting $P$ to $Q$. By recursion, $v(Q_r) = M$ for all $r = 1,...,m$, and in particular, $v(Q) = M$. $\square$

**Proposition 1.** *If $u \in \mathbb{R}^{V^{(0)}}$ and $v \in \mathcal{L}(1,u)$ satisfies $M_1(E)(u) = S_1(E)(v)$, then $v$ satisfies the maximum principle: for every $P \in V^{(1)}$*

$$\min_{V^{(0)}} u \le v(P) \le \max_{V^{(0)}} u\,.$$

*Proof.* We prove for example the second inequality. Suppose that $v(P) = \max v =: M$ for some $P \in V^{(1)} \setminus V^{(0)}$. Using Lemma 7 and Lemma 9, we conclude that $v(Q)(= u(Q)) = M$ for some $Q \in V^{(0)}$. Thus, $M \le \max u$. $\square$

**Theorem 6.** *For every $u \in \mathbb{R}^{V^{(0)}}$ there exists a unique $v \in \mathcal{L}(1,u)$ satisfying $M_1(E)(u) = S_1(E)(v)$, which we will denote by $H_{(1;E)}(u)$.*

*Proof.* Clearly, given $v : V^{(1)} \to \mathbb{R}$, $v \in \mathcal{L}(1,u)$ and satisfies $M_1(E)(u) = S_1(E)(v)$, if and only if $v$ satisfies the equations in (16) and, in addition, the equations $v(P) = u(P)$ for $P \in V^{(0)}$. Thus, we have a linear system with $\#V^{(1)}$ equations and $\#V^{(1)}$ unknowns $v(P)$, $P \in V^{(1)}$. Therefore, we have to prove that the corresponding homogeneous system has no nontrivial solutions. But the homogeneous system is the system corresponding to $u = 0$. By the maximum principle, the unique solution to such a system is the function $v = 0$. $\square$

Clearly, $H_{(1;E)}$ has the following properties:
a) $H_{(1;E)}$ is linear, thus continuous.
b) $H_{(1;E)}(u + c) = H_{(1;E)}(u) + c$ for all $u \in \mathbb{R}^{V^{(0)}}$ and $c \in \mathbb{R}$.

*Remark 4.* The strong maximum principle in general does not hold. For example, in the tree-like Gasket if $E$ is the form defined before Lemma 8, and $u(P_1) = u(P_2) = 0$, then $H_{(1;E)}(u) = 0$ on the 1-cell $V_1$.

We need a further lemma.

**Lemma 10.** *If* $v : V^{(n)} \to \mathbb{R}$ *and* $S_n(E)(v) = 0$*, then* $v$ *is constant on* $V^{(n)}$*.*

*Proof.* We imitate the proof of Lemma 1. Observe that for every $m \in \mathbb{N}$, if $v : V^{(m+1)} \to \mathbb{R}$ has a constant value $c_i$ on every $\psi_i(V^{(m)})$, $i = 1, ..., k$, then $v$ is constant on $V^{(m+1)}$. Indeed, in particular, $v = c_i$ on the 1-cell $V_i$. Since the fractal is connected, $c_i$ is independent of $i$, say $c_i = c$ for each $i$, and since $V^{(m+1)} = \bigcup_{i=1}^{k} \psi_i(V^{(m)})$, $v = c$ on $V^{(m+1)}$ as claimed. It follows that, if $v$ is nonconstant on $V^{(n)}$, then $v \circ \psi_{i_1,i_2,...,i_n}$ is nonconstant on $V^{(0)}$ for some $i_1, i_2, ..., i_n = 1, ..., k$, thus $S_n(E)(v) > 0$. □

**Theorem 7.** $M_1(E) \in \widetilde{\mathcal{D}}$.

*Proof.* By an argument similar to that in Section 2, $M_1(E)$ has a representation as in (15) for suitable coefficients $c'_{j_1,j_2} (= c'_{j_2,j_1})$, $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$. It remains to prove

a) $M_1(E)$ is irreducible.

b) $c'_{j_1,j_2} \geq 0$.

Let us prove a). We have $M_1(E)(u) = S_1(E)(H_{(1;E)}(u))$. If $u$ is nonconstant, so is $H_{(1;E)}(u)$. Hence, in view of Lemma 10, $M_1(E)(u) > 0$. Let us prove b). Let $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$, and $u_{j_1,j_2}$, $v_{j_1,j_2}$ be as in Remark 1, and let $w = H_{(1;E)}(v_{j_1,j_2})$. Since $|w| \in \mathcal{L}(1, u_{j_1,j_2})$, by the definition of $M_1(E)$ we have

$$M_1(E)(u_{j_1,j_2}) \leq S_1(E)(|w|) \leq S_1(E)(w) = M_1(E)(v_{j_1,j_2}), \tag{17}$$

the second inequality being an immediate consequence of the formula

$$(|a| - |b|)^2 \leq (a - b)^2 . \tag{18}$$

Now, b) follows from Remark 1. □

Next, we discuss the problem whether there exist $E \in \widetilde{\mathcal{D}}$ and $\rho > 0$ such that $M_1(E) = \rho E$. In such a case we say that $E$ is an *eigenform* and $\rho$ is its *eigenvalue*. In many fractals an eigenform exists.

*Put* $\overline{E}$ *to be that form having all coefficients equal to* 1.

We have seen that in the Gasket $\overline{E}$ is an eigenform. Although the Vicsek set is less symmetric than the Gasket, it is not difficult to see that its symmetry properties are sufficient to guarantee that $\overline{E}$ is an eigenform (in some sense the kind of connection of two different points $P, P' \in V^{(0)}$ through $V^{(1)}$ is independent of $P, P'$). In the tree-like Gasket a direct calculations shows that every form $E \in \widetilde{\mathcal{D}}$ with $c_{1,3} = 0$ is an eigenform with eigenvalue $\frac{1}{2}$. In fact, let $a = c_{1,2}$, $b = c_{2,3}$. In the definition of $M_1$ we minimize the functional

$$a(u(P_1)-x)^2+a(u(P_2)-x)^2+b(u(P_3)-y)^2+b(u(P_2)-y)^2+b(z-x)^2+a(t-y)^2,$$

where, $v$ being the function to minimize, we put $P_{12} = \psi_1(P_2) = \psi_2(P_1)$, $P_{23} = \psi_3(P_2) = \psi_2(P_3)$, $P_4 = \psi_1(P_3)$, $P_5 = \psi_3(P_1)$, $x = v(P_{12})$, $y = v(P_{23})$, $z = v(P_4)$, $t = v(P_5)$. Clearly, for the minimum $v$ we must have $x = z$, $y = t$, and we have to minimize separately the functions $a(u(P_1)-x)^2+a(u(P_2)-x)^2$ with respect to $x$ and $b(u(P_3)-y)^2+b(u(P_2)-y)^2$ with respect to $y$. Clearly, the result is

$$\frac{1}{2}a(u(P_1)-u(P_2))^2+\frac{1}{2}b(u(P_3)-u(P_2))^2=\frac{1}{2}E(u)\,.$$

For the Snowflake the problem is more complicated. Here, and in general, the map $M_1$, considered as a map from the coefficients of $E$ to the coefficients of $M_1(E)$, is a rational function. This, as the solution $H_{(1;E)}(u)$ of system (16) is given by the quotients of two determinants which are polynomial functions of the coefficients of $E$, and $M_1(E)(u) = S_1(E)(H_{(1;E)}(u))$. So, we cannot expect that $M_1$ is linear, and when ad hoc arguments such as symmetry do not work, the problem of the existence of an eigenform is not trivial. A result of Lindstrøm [9], states that in every nested fractal there exists an eigenform. A nested fractal is defined to be a fractal having properties similar to that of finite ramification, and further the following additional symmetry property:

*If $j_1, j_2 = 1, ..., N$, $j_1 \neq j_2$, then the symmetry $\phi_{j_1,j_2}$ with respect to $W_{j_1,j_2} = \{z : ||z - P_{j_1}|| = ||z - P_{j_2}||\}$, maps $n$-cells to $n$-cells for $n \geq 0$ and any $n$-cell containing elements on both sides of $W_{j_1,j_2}$ is mapped to itself.*

See [9] for the precise definition. It is easy to see that the Gasket, the Vicsek set, and the Snowflake are nested, and the tree-like Gasket is not nested. In particular, on the Snowflake there exists an eigenform. It is not difficult to see that not all fractals have an eigenform. A rather general criterion for the existence of an eigenform also valid for non-nested fractals was given by C. Sabot in his doctoral thesis (1995) (cf. [18]). Improvements of the results of Sabot were given in [13].

*From now on, we assume that in our fractal there exists an eigenform, and $\hat{E}$ will denote a fixed eigenform.*

It is not difficult to see that $\rho < 1$ (see [18]). We can now repeat for $\hat{E}$ the same construction as that for $\overline{E}$ on the Gasket and use the same definitions of $\hat{E}^{\Sigma}_{(n)}$, and of harmonic extension $H_{(\infty;\hat{E})}(u)$. We get

**Theorem 8.** *For every $v : K \to \mathbb{R}$, $\hat{E}^{\Sigma}_{(n)}(v)$ is increasing with respect to $n$. If we put $\hat{E}^{\Sigma}_{(\infty)}(v) = \lim_{n\to\infty} \hat{E}^{\Sigma}_{(n)}(v)$, then $\hat{E}^{\Sigma}_{(\infty)}$ is a good Dirichlet form on $K$.*

*Proof.* We imitate the proof in Section 2. We have to modify slightly the proof of Corollary 1. When there, we stated that the number $\gamma$ is less than 1, we used the strong maximum principle, which, as seen in Remark 4, is no longer valid in this situation. However, a little modification of that argument, using substantially the maximum principle, is still valid. We have to prove that $\mathrm{Osc}_{V_i}(v) < \mathrm{Osc}_{V^{(0)}}(u)$, whenever $u$ is a nonconstant function from $V^{(0)}$ to $\mathbb{R}$ and $v = H_{(1;\hat{E})}(u)$, and $i = 1, ..., k$. To this aim, using the maximum principle, it is sufficient to prove that $v$ cannot attain in the same 1-cell $V_i$ both its maximum and its minimum on $V^{(1)}$. Suppose on the contrary, $\max v = v(\psi_i(P_{j_1}))$, $\min v = v(\psi_i(P_{j_2}))$, $j_1, j_2 = 1, ..., N$. In view of Remark 3, either $i > N$ and $\psi_i(P_{j_1})$ and $\psi_i(P_{j_2})$ are strongly $\hat{E}$-connected, or $i \leq N$ and $\psi_i(P_{j_1})$ and $\psi_i(P_{j_2})$ are both strongly $\hat{E}$-connected to $P_i$. By Lemma 9, in the former case $v(\psi_i(P_{j_1})) = v(\psi_i(P_{j_2}))$, in the latter case $v(\psi_i(P_{j_1})) = v(P_i) = v(\psi_i(P_{j_2}))$. Hence, $\max v = \min v$, and $v$, thus $u$, is constant, a contradiction. □

Note that any positive multiple of an eigenform is an eigenform as well. So, a natural question is: is the eigenform unique up to a multiplicative constant? The answer is yes in several cases, but not always. The first example of nonuniqueness was given by V.Metz in [10], where it was proved that in the Vicsek set, there are infinitely many eigenforms not multiple of each other. A simpler (but less symmetric) example is the tree-like Gasket (see [16]), as we previously saw. We are now going to prove that, independently of the uniqueness, the eigenvalue $\rho$ does not depend on the eigenform, thus it is related to the fractal, and it can be called the *renormalization factor* of the fractal. We need some preparatory considerations.

**Lemma 11.** *For any $E, E' \in \widetilde{\mathcal{D}}$ the ratio*

$$A(u) = \frac{E'(u)}{E(u)}$$

*has positive minimum, which I will denote by $\lambda_-(E, E')$, and maximum, which I will denote by $\lambda_+(E, E')$ on the set of nonconstant $u \in \mathbb{R}^{V^{(0)}}$. They are also, respectively, the minimum and the maximum of $A$ on the set $\mathcal{S}$, defined in (6).*

*Proof.* Since $A(u) = A(\frac{u-u(P_1)}{||u-u(P_1)||})$ for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$, it suffices to observe that $A$ is continuous, thus it has a maximum and a minimum on $\mathcal{S}$. □

*Remark 5.* We clearly have

$$\lambda_-(E, E')E \leq E' \leq \lambda_+(E, E')E$$

for every $E, E' \in \widetilde{\mathcal{D}}$.

We now state some simple properties of $S_n$ and $M_n$ which we will use in the following without explicit mention.

a) $S_n(aE) = aS_n(E)$, $M_n(aE) = aM_n(E)$,

b) $E \le E' \Rightarrow \ S_n(E) \le S_n(E'), M_n(E) \le M_n(E')$,

where $E, E' \in \widetilde{\mathcal{D}}$, $a > 0$. By the expression $E \le E'$ we mean $E(u) \le E'(u)$ for all $u \in \mathbb{R}^{V^{(0)}}$ and similarly for $S_n(E)$ and $M_n(E)$. Note that $E \le E'$ does not amount to $c_{j_1,j_2}(E) \le c_{j_1,j_2}(E')$ for every $j_1, j_2 = 1, ..., N$, $j_1 \ne j_2$; for example, if $N = 3$ and $c_{1,2}(E') = c_{1,3}(E') = 3$, $c_{2,3}(E') = 0$, and $c_{1,2}(E) = c_{1,3}(E) = c_{2,3}(E) = 1$, using the simple inequality $(a+b)^2 \le 2a^2+2b^2$, we have $E \le E'$. We remark that in the previous considerations we did not require that the forms are eigenforms.

**Lemma 12.** *If $E$ and $E'$ are eigenforms in $\widetilde{\mathcal{D}}$, then they have the same eigenvalue.*

*Proof.* Suppose $M_1(E) = \rho E$ and $M_1(E') = \rho' E'$ and prove that $\rho = \rho'$. In fact, we have $aE \le E' \le bE$ for some $a, b > 0$, by Remark 5. Using a) and b), we get $M_1^n(E) = \rho^n E$, $M_1^n(E') = {\rho'}^n E'$, and $a\rho^n E \le {\rho'}^n E' \le b\rho^n E$, for every $n \in \mathbb{N}$. Hence, if $\rho \ne \rho'$, we have that either $E$ or $E'$ is identically 0, contrary to the assumption $E, E' \in \widetilde{\mathcal{D}}$. $\square$

## 4 Main properties of renormalization and harmonic extension

In previous sections, we studied the convergence of $E^{\Sigma}_{(n)}$ when $E$ is an eigenform. We now want to study the corresponding problem when $E$ is any element of $\widetilde{\mathcal{D}}$. The first remark is that, as we know that the eigenvalue $\rho$ is independent of the eigenform, we can well define

$$E^{\Sigma}_{(n)} = \frac{S_n(E)}{\rho^n}$$

for any $E \in \widetilde{\mathcal{D}}$. We are now going to study the following problem: Is the sequence $E^{\Sigma}_{(n)}$ convergent in this more general case? In this case, unlike the case of an eigenform there is no reason that the sequence $E^{\Sigma}_{(n)}$ is increasing. So, the answer is less simple. In order to attack the problem, we need a more careful investigation of the *renormalization operator* $M_n(E)$ and of the harmonic extension on $V^{(n)}$, that, so far, we merely hinted. This is what we will do in this section. We will prove in particular that $M_{n+m}(E) = M_m(M_n(E))$. This will turn out to be a consequence of the nesting property, as, in order to minimize the sum of the copies of $E$ on the $(n+m)$-cells, we can minimize for fixed values on $V^{(m)}$ independently on the different $n$-cells as they only overlap at points in $V^{(m)}$, and then minimize the sum of such minima among the possible values on $V^{(m)}$. The statement of Theorem 9 is thus, in some sense intuitive. However, I will give a complete proof of it. In order to do this, we need two Lemmas. We will use the notation $H_{(m,n;E)}$ for $H_{(m;M_n(E))}$.

**Lemma 13.** *Suppose* $(i_1, ..., i_n) \neq (i'_1, ..., i'_n)$. *Then,*

$$K_{i_1,...,i_n} \cap K_{i'_1,...,i'_n} \subseteq V_{i_1,...,i_n} \cap V_{i'_1,...,i'_n} .$$

*Proof.* Note that $K \subseteq T$, hence $K_{i_1,...,i_n} \cap K_{i'_1,...,i'_n} \subseteq T_{i_1,...,i_n} \cap T_{i'_1,...,i'_n}$. Then, the proof is exactly the same as that in Lemma 3. □

**Lemma 14.** *Suppose* $n \in \mathbb{N}$. *Suppose* $P = \psi_{i_1,...,i_n}(Q) = \psi_{i'_1,...,i'_n}(Q')$ *with* $i_1, ..., i_n, i'_1, ..., i'_n = 1, ..., k$ *and* $Q, Q' \in K$, *and* $(i_1, ..., i_n, Q) \neq (i'_1, ..., i'_n, Q')$. *Then,* $Q, Q' \in V^{(0)}$.

*Proof.* We have $(i_1, ..., i_n) \neq (i'_1, ..., i'_n)$, as in the contrary case, $Q = Q'$. Hence, by Lemma 13, $Q, Q' \in V^{(0)}$. □

**Theorem 9.**

*i) For every* $n \in \mathbb{N}$ *and* $E \in \widetilde{\mathcal{D}}$ *and* $u \in \mathbb{R}^{V^{(0)}}$ *the inf in the definition of* $M_n(E)$ *is in fact a minimum. It is unique and we will denote it by* $H_{(n;E)}(u)$. *Moreover,* $M_n(E) \in \widetilde{\mathcal{D}}$.

*ii) For every* $n, m \in \mathbb{N}$ *and* $E \in \widetilde{\mathcal{D}}$ *and* $u \in \mathbb{R}^{V^{(0)}}$, *on* $V^{(0)}$ *we have*

$$H_{(n+m;E)}(u) \circ \psi_{i_1,...,i_{n+m}} = H_{(n;E)}\big(H_{(m,n;E)}(u) \circ \psi_{i_1,...,i_m}\big) \circ \psi_{i_{m+1},...,i_{m+n}} .$$

*iii) For every* $n, m \in \mathbb{N}$ *and* $E \in \widetilde{\mathcal{D}}$, *we have*

$$M_{n+m}(E) = M_m(M_n(E)) .$$

*Proof.* Suppose i) holds for $n$ and $m$ (for all $E$ and $u$), and prove that the function $\overline{v} : V^{(n+m)} \to \mathbb{R}$ defined by

$$\overline{v}\Big(\psi_{i_1,...,i_m}(Q)\Big) = H_{(n;E)}\big(H_{(m,n;E)}(u) \circ \psi_{i_1,...,i_m}\big)(Q)$$

for $Q \in V^{(n)}$, $i_1, ..., i_m = 1, ..., k$, satisfies

$$\overline{v} \in \mathcal{L}(n+m, u), \quad S_{n+m}(E)(v) \geq M_m\big(M_n(E)\big)(u) \quad \forall v \in \mathcal{L}(n+m, u) , \quad (19)$$

and the equality holds if and only if $v = \overline{v}$. Since we already know that i) holds for $n = 1$, a recursive argument then yields i), ii) and iii). First, note that the definition of $\overline{v}$ is correct, i.e., it does not depend on the representation of $P \in V^{(n+m)}$ as $P = \psi_{i_1,...,i_m}(Q)$ with $i_1, ..., i_m = 1, ..., k$, $Q \in V^{(n)}$. If $P \in V^{(m)}$, then $P = \psi_{i'_1,...,i'_m}(Q')$ for some $i'_1, ..., i'_m = 1, ..., k$, $Q' \in V^{(0)}$. Thus, either $(i_1, ..., i_m, Q) = (i'_1, ..., i'_m, Q')$, then $Q = Q' \in V^{(0)}$, or $(i_1, ..., i_m, Q) \neq (i'_1, ..., i'_m, Q')$, and using Lemma 14 we have $Q \in V^{(0)}$ again. Hence, the definition of $\overline{v}$ gives $\overline{v}(P) = H_{(m,n;E)}(u)(P)$. If, on the contrary, $P \notin V^{(m)}$, in view of Lemma 14 the above representation of $P$ is unique. Next, we prove that $\overline{v} \in \mathcal{L}(n+m,u)$. We just proved that $\overline{v}$ amounts to $H_{(m,n;E)}(u)$ on $V^{(m)} \supseteq V^{(0)}$, which in turn amounts to $u$ on $V^{(0)}$. Finally, if $v \in \mathcal{L}(n+m,u)$, we have

$$S_{n+m}(E)(v) = \sum_{i_1,\dots,i_m=1}^{k} \Big( \sum_{i_{m+1},\dots,i_{m+n}=1}^{k} E(v \circ \psi_{i_1,\dots,i_m} \circ \psi_{i_{m+1},\dots,i_{m+n}}) \Big)$$

$$= \sum_{i_1,\dots,i_m=1}^{k} S_n(E)(v \circ \psi_{i_1,\dots,i_m}) \geq \sum_{i_1,\dots,i_m=1}^{k} M_n(E)\big(v_{|V^{(m)}} \circ \psi_{i_1,\dots,i_m}\big)$$

$$= S_m(M_n(E))\big(v_{|V^{(m)}}\big) \geq M_m\big(M_n(E)\big)(u)\,.$$

Moreover, the first inequality is in fact an equality if and only if

$$v \circ \psi_{i_1,\dots,i_m} = H_{(n;E)}\big(v_{|V^{(m)}} \circ \psi_{i_1,\dots,i_m}\big)$$

on $V^{(n)}$ for all $i_1,\dots,i_m = 1,\dots,k$, the second is an equality if and only if $v_{|V^{(m)}} = H_{(m,n;E)}(u)$, if and only if

$$v \circ \psi_{i_1,\dots,i_m} = H_{(m,n;E)}(u) \circ \psi_{i_1,\dots,i_m}$$

on $V^{(0)}$ for all $i_1,\dots,i_m = 1,\dots,k$. Hence, the equality holds in (19) if and only if $v = \overline{v}$. □

**Corollary 3.** *We have $M_n = M_1^n$.* □

Now, for every $E \in \widetilde{\mathcal{D}}$, we put $\tilde{M}_n(E) := E_{(n)} := \frac{M_n(E)}{\rho^n}$. When $E$ is an eigenform then $M_n(E) = \rho^n E$, so that $E_{(n)} = E$. We easily deduce from Theorem 9 that

$$E_{(n+m)} = (E_{(n)})_{(m)}$$

for all $E \in \widetilde{\mathcal{D}}$ and $n,m \in \mathbb{N}$, thus $\tilde{M}_n = \tilde{M}_1^n$. Clearly, $E \in \widetilde{\mathcal{D}}$ is an eigenform if and only if it is a fixed point of $\tilde{M}_1$. In order to investigate the convergence of $E^{\Sigma}_{(n)}$ we need some information on the convergence of $E_{(n)}$. We will prove in fact that the sequence $E^{\Sigma}_{(n)}$ is $\Gamma$-convergent to $\tilde{E}^{\Sigma}_{(\infty)}$ where $\tilde{E}$ is the limit of $E_{(n)}$. The proof of the convergence of $E_{(n)}$ is the real problem in the proof of $\Gamma$-convergence of $E^{\Sigma}_{(n)}$ when $E$ is not an eigenform. Clearly, when $E$ is an eigenform, $E_{(n)} = E \underset{n\to\infty}{\longrightarrow} E$. In order to prove the convergence of $E_{(n)}$, it will be useful to study the behaviour of $H_{(n;E)}(u)$ on the single $n$-cells, in other words, to study $H_{(n;E)}(u) \circ \psi_{i_1,\dots,i_n}$. In this connection, another consequence of Theorem 9 is that we can split the map $u \mapsto H_{(n;E)}(u) \circ \psi_{i_1,\dots,i_n}$ into the composition of maps like $u \mapsto H_{(1;E)}(u) \circ \psi_i$. We need thus a reformulation of

$H_{(1;E)}(u) \circ \psi_i$ which represents it as a function of $u$. So, define $T_{i;E} : \mathbb{R}^{V^{(0)}} \to \mathbb{R}^{V^{(0)}}$ by

$$T_{i;E}(u) = H_{(1;E)}(u) \circ \psi_i$$

for every $i = 1, ..., k$; $E \in \widetilde{\mathcal{D}}$; $u \in \mathbb{R}^{V^{(0)}}$. Put also $T_{i,n;E} := T_{i;M_n(E)}$. In the previous definitions, we omit $E$ when is clear from the context, and in such a case, we write $T_i, T_{i,n}$. The following properties can be easily verified. We have already discussed some of them in terms of properties of the map $u \mapsto H_{(1;E)}(u)$.

**Proposition 2.** *Let $u \in \mathbb{R}^{V^{(0)}}, E \in \widetilde{\mathcal{D}}$. We have*
*i) $T_i$ is linear.*
*ii) $T_i(u + c) = T_i(u) + c$ if $c$ is constant.*
*iii) $T_i(u)(P_i) = u(P_i)$ for all $i = 1, ..., N$.*
*iv) $T_{i;aE} = T_{i;E}$ if $a > 0$.* □

Note that thanks to iii), for $i = 1, ..., N$, $T_i$ maps the space

$$\Pi_i = \{u \in \mathbb{R}^{V^{(0)}} : u(P_i) = 0\}$$

into itself, and by ii) $T_i$ is completely determined by its values on $\Pi_i \simeq \mathbb{R}^{N-1}$. Thus, $T_i$ can be considered as a linear operator from $\mathbb{R}^{N-1}$ into itself. Another trivial remark is that, by iv), we have $T_{i,M_n(E)} = T_{i,E_{(n)}}$. Putting $n = 1$ in Theorem 9 ii), a simple recursive argument on $m$ yields

**Lemma 15.** *Let $u \in \mathbb{R}^{V^{(0)}}$, $E \in \widetilde{\mathcal{D}}$. Then*

$$H_{(n;E)}(u) \circ \psi_{i_1,...,i_n} = T_{i_n,0} \circ T_{i_{n-1},1} \circ \cdots \circ T_{i_1,n-1}(u) \quad \text{on } V^{(0)} ,$$

*in particular, if $E$ is an eigenform, we have*

$$H_{(n;E)}(u) \circ \psi_{i_1,...,i_n} = T_{i_n} \circ T_{i_{n-1}} \circ \cdots \circ T_{i_1}(u) \quad \text{on } V^{(0)}. \square$$

**Corollary 4.** *Let $E \in \widetilde{\mathcal{D}}$, $u \in \mathbb{R}^{V^{(0)}}$, $Q \in V^{(n)}$. Then*

$$\min_{V^{(0)}} u \le H_{(n;E)}(u)(Q) \le \max_{V^{(0)}} u \,.$$

*Proof.* We have $\min_{V^{(0)}} u \le T_{i;E}(u) \le \max_{V^{(0)}} u$ for every $i = 1, ..., k$. We conclude by a recursive argument. □

So far, I discussed about the convergence of sequences in $\widetilde{\mathcal{D}}$, but I did not specify by what a sense I mean the convergence. It will be better to consider, a priori, the convergence on $\mathcal{D}$ instead of on $\widetilde{\mathcal{D}}$, as $\mathcal{D}$ is closed. We can define a norm $|| \; ||$ on the linear space generated by $\mathcal{D}$ as $||E|| = \sup_{u \in \mathcal{S}} |E(u)|$. Note that if $||E|| = 0$, then as $E$ is 2-homogeneous and satisfies $E(u) = E(u - u(P_1))$, it follows $E(u) = 0$ for all $u \in \mathbb{R}^{V^{(0)}}$, so that $|| \; ||$ is in fact a norm. We have:

**Lemma 16.** *Given $E_n, E \in \mathcal{D}$, the following properties are equivalent*

*a)* $E_n \underset{n\to\infty}{\longrightarrow} E$ *pointwise,*

*b)* $E_n \underset{n\to\infty}{\longrightarrow} E$ *uniformly on the compact subsets of* $\mathbb{R}^{V^{(0)}}$*,*

*c)* $E_n \underset{n\to\infty}{\longrightarrow} E$ *in the norm* $||\ ||$*,*

*d)* $c_{j_1,j_2}(E_n) \underset{n\to\infty}{\longrightarrow} c_{j_1,j_2}(E)$ *for all* $j_1, j_2 = 1, ..., N$*,* $j_1 \neq j_2$*.*

*Proof.* Clearly, we have $d) \Rightarrow b) \Rightarrow a)$. Also, $a) \Rightarrow d)$ by Remark 1. Since the set $\mathcal{S}$ is compact, we have $b) \Rightarrow c)$. Finally, as $E(u) = ||u - u(P_1)||^2 E\Big(\frac{u-u(P_1)}{||u-u(P_1)||}\Big)$ for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$, $c) \Rightarrow a)$. $\square$

So, the convergence in $\mathcal{D}$ (a fortiori in $\widetilde{\mathcal{D}}$) will be meant to be in one of the four equivalent formulations in Lemma 16. Also, we will consider on $\mathcal{D}$ and on $\widetilde{\mathcal{D}}$ the topology induced by such a convergence. In this way the main functions we defined on $\widetilde{\mathcal{D}}$ are continuous.

**Lemma 17.** *We have*

*i) The map from* $\widetilde{\mathcal{D}} \times \mathbb{R}^{V^{(0)}}$ *to* $\mathbb{R}^{V^{(1)}}$ *defined by* $(E, u) \mapsto H_{(1;E)}(u)$*, is continuous.*

*ii) The map from* $\widetilde{\mathcal{D}} \times \mathbb{R}^{V^{(0)}}$ *to* $\mathbb{R}$ *defined by* $(E, u) \mapsto M_1(E)(u)$ *is continuous.*

*iii)* $M_n$ *is continuous from* $\widetilde{\mathcal{D}}$ *to* $\widetilde{\mathcal{D}}$*.*

*iv)* $\lambda_+$ *and* $\lambda_-$ *are continuous from* $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$ *to* $]0, +\infty[$*.*

*Proof.* Prove i). Suppose $E_h, E \in \widetilde{\mathcal{D}}$, $u_h, u \in \mathbb{R}^{V^{(0)}}$, $E_h \underset{h\to\infty}{\longrightarrow} E$, $u_h \underset{h\to\infty}{\longrightarrow} u$, and prove that

$$H_{(1;E_h)}(u_h) \underset{h\to\infty}{\longrightarrow} H_{(1;E)}(u)\,. \tag{20}$$

By the maximum principle we have $\min\limits_{V^{(0)}} u_h \leq H_{(1;E_h)}(u_h) \leq \max\limits_{V^{(0)}} u_h$. Thus, the functions $H_{(1;E_h)}(u_h)$ are uniformly bounded and if (20) does not hold, we have

$$H_{(1;E_{h_l})}(u_{h_l}) \underset{l\to\infty}{\longrightarrow} v$$

with $v \neq H_{(1;E)}(u)$, for some strictly increasing sequence of naturals $h_l$. Thus, $v$ satisfies (16), and $v = H_{(1;E)}(u)$, contrary to our assumption. ii) easily follows from i), in view of the formula $M_1(E)(u) = S_1(E)\big(H_{(1;E)}(u)\big)$. iii) is an immediate consequence of ii) and of the formula $M_n(E) = M_1^n(E)$. We now prove iv). Suppose $E_h \underset{h\to\infty}{\longrightarrow} E$, $E'_h \underset{h\to\infty}{\longrightarrow} E'$. Given a nonconstant $u \in \mathbb{R}^{V^{(0)}}$, let $\alpha = E(u)$ and let $\overline{h} \in \mathbb{N}$ be such that $E_h(u) \geq \frac{\alpha}{2}$ for $h \geq \overline{h}$. By simple calculations, for every $h \geq \overline{h}$ and $u \in \mathcal{S}$ we get

$$\Big|\frac{E'_h(u)}{E_h(u)} - \frac{E'(u)}{E(u)}\Big| \le \frac{2}{\alpha^2}\big|E'_h(u)E(u) - E_h(u)E'(u)\big|$$

$$\le \frac{2}{\alpha^2}\Big(||E'_h||\ ||E_h - E|| + ||E_h||\ ||E'_h - E'||\Big) \underset{h\to\infty}{\longrightarrow} 0\,,$$

so that iv) easily follows. □

**Corollary 5.** *If* $E, E' \in \widetilde{\mathcal{D}}$ *and* $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$*, then* $E'$ *is an eigenform.*

*Proof.* Since $E' = \lim_{n\to\infty} \big(\tilde{M}_1\big)^n(E)$, and $\tilde{M}_1$ is continuous, then $E'$ is a fixed point of $\tilde{M}_1$. □

Lemma 17 and Corollary 5 will be used without mention in the following. Another equivalent way of expressing the convergence in $\widetilde{\mathcal{D}}$ is given by the following corollary.

**Corollary 6.** *Given* $E_n, E \in \widetilde{\mathcal{D}}$*, then* $E_n \underset{n\to\infty}{\longrightarrow} E \iff \lambda_\pm(E, E_n) \underset{n\to\infty}{\longrightarrow} 1.$

*Proof.* Part "if" follows from Remark 5. Part "only if" follows from Lemma 17 iv. □

In some sense, the functions $\lambda_\pm$ provide a way of measuring how much far two different $E, E' \in \widetilde{\mathcal{D}}$ are. We are so lead to give the following definition.

*Given* $E, E' \in \widetilde{\mathcal{D}}$ *let*

$$\lambda(E, E') = \ln(\lambda_+(E, E')) - \ln(\lambda_-(E, E')).$$

$\lambda$ is a particular case of Hilbert's projective metric. For the theory of Hilbert's projective metric see for example [15]. We have in fact, as can be easily verified, that $\lambda$ is a semimetric on $\widetilde{\mathcal{D}}$, in the sense that it has all properties of a metric but the property that the distance is 0 only if the two elements are the same. More precisely, we have $\lambda(E, E') = 0$ if and only if $E'$ is a (positive) multiple of $E$. In particular, $E \in \widetilde{\mathcal{D}}$ is an eigenform if and only if $\lambda(E, E_{(1)}) = 0$. Also, we clearly have $\lambda(aE, bE') = \lambda(E, E')$ for every $a, b > 0$. Thus, $\lambda$ induces a metric on the projective space $pr(\widetilde{\mathcal{D}})$ generated by $\widetilde{\mathcal{D}}$, that is the space of equivalence classes on $\widetilde{\mathcal{D}}$, mod the relation that identifies $E$ to $aE$ for $a > 0$, $E \in \widetilde{\mathcal{D}}$. I will now describe some simple properties of $\lambda$. While the previous results, in Sections 1 to 4 can be found (possibly in a slightly different form) in textbooks on this topic, for example [6], the following properties of $\lambda$ can be found in at least one among [12], [16], [18] (cf. also [11]). I here follow [16].

**Lemma 18.** *Given* $E, E' \in \widetilde{\mathcal{D}}$ *we have*

*i)* $\lambda_-(M_1(E), M_1(E')) = \lambda_-(E_{(1)}, E'_{(1)}) \ge \lambda_-(E, E')$

*ii)* $\lambda_+(M_1(E), M_1(E')) = \lambda_+(E_{(1)}, E'_{(1)}) \le \lambda_+(E, E')$

*iii)* $\lambda(M_1(E), M_1(E')) = \lambda(E_{(1)}, E'_{(1)}) \le \lambda(E, E')$.

*Proof.* In i), ii), iii), the equalities are trivial, so we have to prove the inequalities. By Remark 5 we have $\lambda_-(E,E')M_1(E) \le M_1(E') \le \lambda_+(E,E')M_1(E)$, hence,

$$\lambda_-(E,E') \le \frac{M_1(E')(u)}{M_1(E)(u)} \le \lambda_+(E,E')$$

for every nonconstant $u \in \mathbb{R}^{V^{(0)}}$, and i) and ii) follow at once, and iii) is an immediate consequence of i) and ii). □

Now, since $E_{(1)}$ is a multiple of $M_1(E)$ for $E \in \widetilde{\mathcal{D}}$, in view of Lemma 18, the map $\tilde{M}_1$ is a weak contraction on $pr(\widetilde{\mathcal{D}})$ with respect to $\lambda$. This fact suggests that a way for proving that the iterated $\tilde{M}_n = \tilde{M}_1^n$ converges could be to try to prove that the above considered map is in fact a strong contraction. Unfortunately, this is not true in general, as we saw that we can have two different normalized eigenforms, which correspond to two different (on $pr(\widetilde{\mathcal{D}})$) fixed points of $\tilde{M}_1$. However, we can modify such an argument in this way: In order to prove that the sequence $E_{(n)}$ tends to some eigenform $\tilde{E}$, since the distance $\lambda$ between $\tilde{E}$ and $E_{(n)}$ is decreasing in view of the previous lemma, we can try to prove that it is not eventually constant, so that we can hope that it tends to 0. We will use this argument on the Gasket in next section. We are thus lead to investigate the cases in which in Lemma 18 iii) we have the equality. For the moment, however, let us discuss some simple consequences of Lemma 18. Given an eigenform $\tilde{E}$ and real numbers $a, b$ with $b \ge a > 0$, let us put

$$U_{a,b,\tilde{E}}$$

$$= \big\{E \in \widetilde{\mathcal{D}} : a\tilde{E} \le E \le b\tilde{E}\big\}\Big( = \big\{E \in \widetilde{\mathcal{D}} : a \le \lambda_-(\tilde{E},E), \lambda_+(\tilde{E},E) \le b\big\}\Big)$$

and write $U_{a,b}$ for $U_{a,b,\hat{E}}$. Since $\hat{E}$ is an eigenform and thus $\hat{E}_{(1)} = \hat{E}$, it immediately follows from Lemma 18 that $\tilde{M}_1$ maps $U_{a,b}$ into itself. Thus if $E \in U_{a,b}$, then every $E_{(n)}$ lies in $U_{a,b}$. Note that every $E \in \widetilde{\mathcal{D}}$ belongs to $U_{a,b}$ if, for example $a = \lambda_-(\hat{E},E)$, $b = \lambda_+(\hat{E},E)$. Moreover, $U_{a,b}$ is (sequentially) compact. We have in fact:

**Lemma 19.** *Every sequence in $U_{a,b}$ has a subsequence convergent to some element of $U_{a,b}$.*

*Proof.* Let $E_n$ be a sequence in $U_{a,b}$. Because of Remark 1, the coefficients $c_{j_1,j_2}(E_n)$ are estimated by $\frac{b}{4}\hat{E}(v_{j_1,j_2})$. Thus, $E_n$ has a subsequence convergent to some functional $E$ and we immediately see that $E \in \mathcal{D}$. Moreover, we clearly have $a\hat{E} \le E \le b\hat{E}$. Thus, if $u \in \mathbb{R}^{V^{(0)}}$ is nonconstant we have $E(u) \ge a\hat{E}(u) > 0$, so that $E \in \widetilde{\mathcal{D}}$. Moreover, $E \in U_{a,b}$. □

**Corollary 7.** *For every $E \in \widetilde{\mathcal{D}}$ there exists a strictly increasing sequence of naturals $n_h$ and $E' \in \widetilde{\mathcal{D}}$ such that $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$.* □

The problem is thus to prove that all the sequence $E_{(n)}$ converges to the same limit $E'$. The use of $U_{a,b}$ is the point in which we need the existence of an eigenform. Lemma 19, in fact, states that in some sense the sequence $E_{(n)}$ is bounded with respect to $\lambda$. Another consequence of Lemma 18 is the following.

**Corollary 8.** *Given $E, E' \in \widetilde{\mathcal{D}}$, put*

$$\lambda_{\pm,n} = \lambda_{\pm,n}(E,E') = \lambda_\pm(E_{(n)}, E'_{(n)}) \quad \lambda_n = \lambda_n(E,E') = \lambda(E_{(n)}, E'_{(n)})\,.$$

*Then*

*i) $\lambda_{+,n}$ is decreasing and $\lambda_{-,n}$ is increasing, thus $\lambda_n$ is decreasing*

*ii) If we set $\lambda_\pm = \lim\limits_{n\to\infty} \lambda_{\pm,n}$ we have $0 < \lambda_- \le \lambda_+ < +\infty$.*

*Proof.* Since $E_{(n+1)} = (E_{(n)})_{(1)}$, i) is an immediate consequence of Lemma 18, putting there $(E_{(n)})$ in place of $E$ and $(E'_{(n)})$ in place of $E'$. For ii), note that $0 < \lambda_{-,0} \le \lambda_{-,n} \le \lambda_{+,n} \le \lambda_{+,0} < +\infty$. □

**Corollary 9.** *If $E, E' \in \widetilde{\mathcal{D}}$ and $E'$ is an eigenform and $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$ for some strictly increasing sequence of naturals $n_h$, then $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$.*

*Proof.* By Corollary 6, $\lambda_\pm(E', E_{(n_h)}) \underset{h\to\infty}{\longrightarrow} 1$. On the other hand, since $E'_{(n)} = E'$, there exists $\lim\limits_{n\to\infty} \lambda_\pm(E', E_{(n)}) = 1$. By Corollary 6 again, $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$. □

We have just seen that $\lambda_{+,n} \le \lambda_{+,0}$ and $\lambda_{-,n} \ge \lambda_{-,0}$. In order to know whether the operator $\tilde{M}_n$ contracts the distance $\lambda$ for some $n$, we need to know when such inequalities are in fact equalities. To this aim, we study the set of functions in $\mathbb{R}^{V^{(0)}}$, which are in some sense extrema for the ratio $\frac{E'}{E}$. For $E, E' \in \widetilde{\mathcal{D}}$ put

$$(A^\pm =) A^\pm(E,E') = \left\{u \in \mathbb{R}^{V^{(0)}} : E'(u) = \lambda_\pm(E,E')E(u)\right\}.$$

By definition such sets include the constant functions. Since in some cases we need to use only nonconstant functions, put $(\tilde{A}^\pm =) \tilde{A}^\pm(E,E')$ to be the set of the functions in $A^\pm(E,E')$ which are not constant. Put also

$$(A^{\pm,n} =) A^{\pm,n}(E,E') = A^\pm(M_n(E), M_n(E'))\big(= A^\pm(E_{(n)}, E'_{(n)})\big)\big),$$

$$(\tilde{A}^{\pm,n} =) \tilde{A}^{\pm,n}(E,E') = \tilde{A}^\pm(M_n(E), M_n(E'))\big(= \tilde{A}^\pm(E_{(n)}, E'_{(n)})\big)\big).$$

**Proposition 3.** *Let $E, E' \in \widetilde{\mathcal{D}}$. Then*
*i) $A^{\pm}$ is closed.*
*ii) $u \in A^{\pm} \Rightarrow c_1 u + c_2 \in A^{\pm} \quad \forall c_1, c_2 \in \mathbb{R}$.*
*iii) $E'$ is a multiple of $E \iff \tilde{A}^{+} \cap \tilde{A}^{-} \neq \emptyset$.*
*iv) $\tilde{A}^{\pm} \neq \emptyset$.*

Note that, if $\lambda_{\pm,n} = \lambda_{\pm,0}$, then

$$A^{\pm,n}(E, E') = \left\{ u \in \mathbb{R}^{V^{(0)}} : M_n(E')(u) = \lambda_{\pm,0} M_n(E)(u) \right\}.$$

Now, we can state the following result.

**Lemma 20.** *Let $E$ and $E'$ and $\lambda_{\pm,n}$ be as in Corollary 8. If we have $\lambda_{\pm,n} = \lambda_{\pm,0}$, then for every $u \in A^{\pm,n}$ we have*

$$H_{(n;E)}(u) \circ \psi_{i_1,\ldots,i_n} = H_{(n;E')}(u) \circ \psi_{i_1,\ldots,i_n} \in A^{\pm}$$

*for all $i_1, \ldots, i_n = 1, \ldots, k$.*

*Proof.* Let $u \in A^{-,n}$. Then

$$M_n(E')(u) = S_n(E')\big(H_{(n;E')}(u)\big) =$$

$$\sum_{i_1,\ldots,i_n=1}^{k} E'\big(H_{(n;E')}(u) \circ \psi_{i_1,\ldots,i_n}\big) \geq \sum_{i_1,\ldots,i_n=1}^{k} \lambda_{-,0} E\big(H_{(n;E')}(u) \circ \psi_{i_1,\ldots,i_n}\big)$$

$$= \lambda_{-,0} S_n(E)\big(H_{(n;E')}(u)\big) \geq \lambda_{-,0} M_n(E)(u) = M_n(E')(u)$$

so that the two inequalities are in fact equalities. From the fact that the first inequality is an equality we deduce $H_{(n;E')}(u) \circ \psi_{i_1,\ldots,i_n} \in A^{-}$, and from the fact that the second inequality is an equality we deduce $H_{(n;E)}(u) \circ \psi_{i_1,\ldots,i_n} = H_{(n;E')}(u) \circ \psi_{i_1,\ldots,i_n}$, for all $i_1, \ldots, i_n = 1, \ldots, k$. We have proved the Lemma for the case where $\pm$ is $-$. We can proceed similarly in the case where $\pm$ is $+$. □

**Corollary 10.** *In the same hypotheses as in Lemma 20, for every $m$ with $0 \leq m \leq n$ we have*

$$\lambda_{\pm,n} = \lambda_{\pm,m} = \lambda_{\pm,0} = \lambda_{\pm,n-m}(M_m(E), M_m(E')) = \lambda_{\pm,0}(M_m(E), M_m(E')), \tag{21}$$

*and for every $u \in A^{\pm,n}$*

$$H_{(n-m,m;E)}(u) \circ \psi_{i_1,\ldots,i_{n-m}} = H_{(n-m,m;E')}(u) \circ \psi_{i_1,\ldots,i_{n-m}} \in A^{\pm,m} \tag{22}$$

*for all $i_1, \ldots, i_{n-m} = 1, \ldots, k$.*

*Proof.* From Corollary 8, (21) follows at once. Thus, (22) follows from Lemma 20 and the fact that

$$u \in A^{\pm}\Big(M_{n-m}\big(M_m(E)\big), M_{n-m}\big(M_m(E')\big)\Big). \qquad \square$$

Note that the first equality in (22) for every $i_1, ..., i_{n-m} = 1, ..., k$, simply amounts to $H_{(n-m,m;E)}(u) = H_{(n-m,m;E')}(u)$. Roughly speaking, whenever we have $\lambda_{\pm,n} = \lambda_{\pm,0}$, every function $u \in A^{\pm,n}$ produces, via its harmonic extension, functions in $A^{\pm,m}$ for all $m \le n$, at every $(n-m)$-cell, in particular, for $m = 0$, functions in $A^{\pm}$, at every $n$-cell.

*Remark 6.* Note that, given $E, E' \in \widetilde{\mathcal{D}}$, if $\lambda(E_{(n)}, E'_{(n)}) = \lambda(E, E')$ then $\lambda_{\pm}(E_{(n)}, E'_{(n)}) = \lambda_{\pm}(E, E')$.

## 5 Homogenization on the Gasket

In this section we assume that the fractal is the Gasket. We first prove that for every $E \in \widetilde{\mathcal{D}}$, the sequence $E_{(n)}$ converges, then we prove that the sequence $E^{\Sigma}_{(n)}$ is $\Gamma$-convergent. We call this phenomenon *homogenization*. Analogous convergence results also hold for general fractals, but I prefer first to illustrate the process in the Gasket, because the proof is more natural and so can be better understood. A typical property of the Gasket (and of other, but not of all, fractals), is that the limit form of $E_{(n)}$ is a multiple of $\overline{E}$. The proof of the convergence of $E_{(n)}$ on the Gasket presented in this section has not been published, but the idea is sketched in the introduction of [16]. The proof in general fractals presented in Section 6 follows more or less the approach of [16]. In Section 4 we studied what happens if $\lambda_{\pm}(E_{(n)}, E'_{(n)}) = \lambda_{\pm}(E, E')$ in the general case $E, E' \in \widetilde{\mathcal{D}}$. Put now $E = \overline{E}$, so that $E_{(n)} = \overline{E}$, and put $E$ in place of $E'$. Since $\overline{E}$ is an eigenform, then (22) yields $H_{(n-m;\overline{E})}(u) \circ \psi_{i_1,...,i_{n-m}} \in A^{\pm,m}$ if $u \in A^{\pm,n}$. Hence, in view of Lemma 15, we get

**Proposition 4.** *If $E \in \widetilde{\mathcal{D}}$ and $\lambda_{\pm}(\overline{E}, E_{(n)}) = \lambda_{\pm}(\overline{E}, E)$ and $u \in A^{\pm}(\overline{E}, E_{(n)})$, then $T^n_{i;\overline{E}}(u) \in A^{\pm}(\overline{E}, E)$ for every $i = 1, 2, 3$.*

Let $T_i := T_{i;\overline{E}}$ in the rest of this section. The plan of the proof of the convergence of $E_{(n)}$ is more or less the following. First, we will prove that if such a convergence does not take place for some $E \in \widetilde{\mathcal{D}}$, then we have $\lambda_{\pm}(\overline{E}, E_{(n)}) = \lambda_{\pm}(\overline{E}, E)$ for some $E \in \widetilde{\mathcal{D}}$; next, we will use Prop. 4 to deduce that $\tilde{A}^+$ and $\tilde{A}^-$ contain some eigenvectors of $T_i$, obtained as a limit of $T^n_i$ and, finally we will prove that these eigenvectors are the same, so that by Prop. 3, $E$ is a multiple of $\overline{E}$, and thus $E_{(n)} = E$. In order to prove the first step, I will use the following notation. By Corollary 8, $\lambda(\overline{E}, E_{(n)})$ is decreasing in $n$, so that $\lambda(\overline{E}, E_{(n)}) \le \lambda(\overline{E}, E)$ for each $n$. I will say that $E \in \widetilde{\mathcal{D}}$ *approaches* $\overline{E}$ if $\lambda(\overline{E}, E_{(n)}) < \lambda(\overline{E}, E)$ for some, thus for sufficiently large, $n$. I will say that $E$ *goes to* $\overline{E}$ if $E_{(n)}$ tends, as $n \to \infty$, to a multiple of $\overline{E}$. Then

**Lemma 21.** *If every $E \in \widetilde{\mathcal{D}}$ which is not a multiple of $\overline{E}$ approaches $\overline{E}$, then every $E \in \widetilde{\mathcal{D}}$ goes to $\overline{E}$.*

*Proof.* Let $E \in \widetilde{\mathcal{D}}$. By Corollary 7 there exist a strictly increasing sequence of naturals $n_h$ and $E' \in \widetilde{\mathcal{D}}$ such that $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$. For every $m \in \mathbb{N}$ we have

$$\lambda(\overline{E}, E'_{(m)}) = \lambda\big(\overline{E}, \lim_{h\to\infty} (E_{(n_h)})_{(m)}\big) = \lambda(\overline{E}, \lim_{h\to\infty} E_{(m+n_h)})$$

$$= \lim_{h\to\infty} \lambda(\overline{E}, E_{(m+n_h)}) = \lim_{n\to\infty} \lambda(\overline{E}, E_{(n)}) \,,$$

the last equality depending on the fact that the last limit exists. Thus, $E'$ does not approach $\overline{E}$ and by hypothesis, $E' = a\overline{E}$ for some positive $a$, in particular it is an eigenform. By Corollary 9, $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$. $\square$

In order to investigate the convergence of $T_j$, I first introduce the problem by some preliminary considerations. If $u \in \mathbb{R}^{V^{(0)}}$ we identify $u$ with a vector of $\mathbb{R}^3$ by putting $u = (u(P_1), u(P_2), u(P_3))$. By this identification, $T_j$ is a linear operator from $\mathbb{R}^3$ into itself. Suppose for example $j = 3$. Then $T_3$ maps $\mathbb{R}^2 \times \{0\}$ into itself. We have $T_3(x, y, 0) = \Big(H_{(1,\overline{E})}(x, y, 0)(\psi_3(P_i))_{i=1,2,3}\Big)$ $= (\frac{2}{5}x + \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y, 0)$ (see solution of (3)). Hence, putting $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, we get $T_3(e_1) = (\frac{2}{5}, \frac{1}{5}, 0)$, $T_3(e_2) = (\frac{1}{5}, \frac{2}{5}, 0)$. The really important point in these formulas is that

$$T_3(e_1) = (a, b, 0), T_3(e_2) = (b, a, 0) \quad \text{with } a, b > 0\,. \tag{23}$$

This can be deduced by the symmetry of $P_1$ and $P_2$ and by the strong maximum principle, with no explicit calculations. On the base of the form of $T_3$ we see that $T_3$ maps the positive cone $D_3 := \{v \in \mathbb{R}^3 : v_1 \geq 0, v_2 \geq 0, v_3 = 0, v \neq (0,0,0)\}$ into the cone $D_3' := \{v \in \mathbb{R}^3 : v_1 > 0, v_2 > 0, v_3 = 0\}$. Suppose now that $\lambda(\overline{E}, E_{(n)}) = \lambda(\overline{E}, E)$, and take $u_n \in \tilde{A}^{\pm}(\overline{E}, E_{(n)})$. Suppose for example it attains its minimum at $P_3$. By Prop. 3 ii), we can and do assume $u_n(P_3) = 0$, so that $u_n \in D_3$. We have $w_n := T_3^n(u_n) \in \tilde{A}^{\pm}$. Clearly $w_n \underset{n\to\infty}{\longrightarrow} 0$, but for our considerations it is specially interesting to know what is the limit of the normalized vector $v_n := \frac{w_n}{||w_n||}$ (if it exists). Since $T_3$ in some sense has the effect of mixing the first two components in a symmetric way, we can expect that $v_n \underset{n\to\infty}{\longrightarrow} \frac{\sqrt{2}}{2}(1, 1, 0)$; however, since the point $u_n$ depends on $n$, this cannot be simply proved on the base on the convergence of the iterated, but we need a sort of uniform convergence. We now come to give the precise statement of the convergence of $T_j^n$. Let

$$D_j := \{v \in \mathbb{R}^3 : v_l \geq 0 \text{ for } l \neq j, v_j = 0, v \neq (0,0,0)\},$$

$$D_j' := \{v \in \mathbb{R}^3 : v_l > 0 \text{ for } l \neq j, v_j = 0, \}\,.$$

Let $\overline{v}_1 = \big(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big)$, $\overline{v}_2 = \big(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\big)$, $\overline{v}_3 = \big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\big)$. Let $\widehat{T_j^n}(v) = \frac{T_j^n(v)}{||T_j^n(v)||}$ when $T_j^n(v) \neq 0$. Let $q(v) = \dfrac{\max v_i}{\min v_i}$, where the maximum and the minimum are taken over $i = 1, 2, 3$ with $i \neq j$, when $v \in D_j'$. Then

**Lemma 22.** *We have* $\widehat{T_j^n}(D_j) \underset{n\to\infty}{\longrightarrow} \overline{v}_j$, *in the sense that*

$$\sup_{v\in D_j} \Big( q\big(T_j^n(v)\big)\Big) \underset{n\to\infty}{\longrightarrow} 1\,.$$

*Proof.* We suppose for example $j = 3$, the other cases being analogous. Define functions $q_1, q_2$ on $D_3'$ by $q_1(v) = \frac{v_2}{v_1}$, $q_2(v) = \frac{v_1}{v_2}$. Thus

$$q = \max\{q_1, q_2\}\,.$$

Note that $q_1$, $q_2$, and thus $q$, are continuous with values in $]0, +\infty[$ . By (23) we have for $v \in D_3'$

$$q_1(T_3(v)) = \frac{\frac{1}{5}v_1 + \frac{2}{5}v_2}{\frac{2}{5}v_1 + \frac{1}{5}v_2} = \frac{2q_1(v) + 1}{q_1(v) + 2}$$

and a similar formula holds for $q_2(T_3(v))$. In other words, for $i = 1, 2$, we have $q_i(T_3(v)) = f(q_i(v))$ where

$$f(t) = \frac{2t + 1}{t + 2}\,.$$

Note that $f$ satisfies

a) $f(1) = 1$.
b) $f$ is strictly increasing on $]0, +\infty[$.
c) $f(t) < t$ for every $t > 1$.

We deduce $q(T_3(v)) = f(q(v))$. Also, for every $t \geq 1$,

$$f^n(t) \underset{n\to\infty}{\longrightarrow} 1\,. \tag{24}$$

Moreover, the convergence in (24) is uniform on every interval of the form $[1, c]$ with $c > 1$, since, by b) we have $1 \leq f^n(t) \leq f^n(c)$ for $t \in [1, c]$. Now, $q$ has a maximum $M \geq 1$ on $T_3(D_3)$ as $q$ is continuous and positively 0-homogeneous, and the set of unit vectors in $D_3$, so its image via $T_3$, is compact. Since $q(T_3^n(v)) = f^n(q(v))$, we have

$$\sup_{v\in D_3} q\big(T_3^n(v)\big) \underset{n\to\infty}{\longrightarrow} 1\,.\square$$

**Corollary 11.** *Given a strictly increasing sequence $n_h$ of naturals, and $v_h \in \widehat{T_j^{n_h}}(D_j)$, we have $v_h \underset{h\to\infty}{\longrightarrow} \overline{v}_j$ .*

*Proof.* Let for example $j = 3$. By Lemma 22 we have $q(v_h) \underset{h\to\infty}{\longrightarrow} 1$. If $w$ is a limit point of $v_h$, then $w \in D_3'$, as, in the contrary case, a subsequence of $q(v_h)$ tends to $+\infty$. Hence, by continuity, $q(w) = 1$, which implies $w = \overline{v}_3$. Since $\overline{v}_3$ is the unique limit point of $v_h$, then $v_h \underset{h\to\infty}{\longrightarrow} \overline{v}_3$. $\square$

**Theorem 10.** *Every $E \in \widetilde{\mathcal{D}}$ goes to $\overline{E}$.*

*Proof.* By Lemma 21 it suffices to prove that given $E \in \widetilde{\mathcal{D}}$ that does not approach $\overline{E}$, then $E$ is a multiple of $\overline{E}$. Take $u_n \in \tilde{A}^+(\overline{E}, E_{(n)})$. Clearly, there exists a strictly increasing sequence of naturals $n_h$ such that all $u_{n_h}$ attain their minima at the same point $P_{j_1}$ and their maxima at the same point $P_{j_2}$, with $j_1 \neq j_2$ . By Prop. 3 ii we also have $w_{n_h} := u_{n_h} - u_{n_h}(P_{j_1}) \in \tilde{A}^+(\overline{E}, E_{(n_h)})$, $w'_{n_h} := u_{n_h}(P_{j_2}) - u_{n_h} \in \tilde{A}^+(\overline{E}, E_{(n_h)})$. Moreover, $w_{n_h} \in D_{j_1}$, $w'_{n_h} \in D_{j_2}$. Thus, $v_{n_h} := \widehat{T_{j_1}^{n_h}}(w_{n_h}) \in A^+$ by Prop. 4, and by Corollary 11, $v_{n_h} \underset{h\to\infty}{\longrightarrow} \overline{v}_{j_1}$, hence $\overline{v}_{j_1} \in \tilde{A}^+$. By proceeding in a similar way, $v'_{n_h} := \widehat{T_{j_2}^{n_h}}(w'_{n_h}) \in A^+$, and $v'_{n_h} \underset{h\to\infty}{\longrightarrow} \overline{v}_{j_2} \in \tilde{A}^+$. By the same argument there exist $j'_1, j'_2$, with $j'_1 \neq j'_2$ such that $\overline{v}_{j'_1}, \overline{v}_{j'_2} \in A^-$. Since we have only three points in $V^{(0)}$ there exists $\overline{v}_j \in \tilde{A}^+ \cap \tilde{A}^-$, and, by Prop. 3 iii, $E$ is a multiple of $\overline{E}$. □

*Remark 7.* The argument in the proof of Theorem 10 heavily relies on the fact that $N = 3$. Aiming to possible extension of the proof to more general fractals, it could be useful to observe that a modification of that argument does not use $N = 3$, but uses the strong symmetry of the Gasket, which implies in particular that $\overline{v}_j$ is symmetric with respect to the components different from $j$. We saw in proof of Theorem 10 that there exists $j(= j(0)) = 1, 2, 3$ such that $\overline{v}_j \in A^+(\overline{E}, E)$. On the other hand, clearly, $E_{(n)}$ does not approach $\overline{E}$, so that we can apply the same argument to $E_{(n)}$ and there exists $j(n) = 1, 2, 3$ such that $\overline{v}_{j(n)} \in A^+(\overline{E}, E_{(n)})$. We can now repeat the previous argument. Let $n_h$ be a strictly increasing sequence of naturals and let $\overline{j} = 1, 2, 3$ be such that $j(n_h) = \overline{j}$ for all $h$. Then, by Prop. 4, $\widehat{T_{\overline{j}}^{n_h}}(\overline{v}_{\overline{j}}) \in A^+$, and, since $\overline{v}_{\overline{j}} \in D_{\overline{j}}$, by Corollary 11

$$\widehat{T_{\overline{j}}^{n_h}}(\overline{v}_{\overline{j}}) \underset{h\to\infty}{\longrightarrow} \overline{v}_{\overline{j}} \in \tilde{A}^+ .$$

On the other hand, for $j \neq \overline{j}$, $\tilde{v}_{\overline{j}} := (1, 1, 1) - \sqrt{2}\,\overline{v}_{\overline{j}} \in D_j \cap A^+$, hence

$$\widehat{T_j^{n_h}}(\tilde{v}_{\overline{j}}) \underset{h\to\infty}{\longrightarrow} \overline{v}_j \in \tilde{A}^+ .$$

In conclusion, $\overline{v}_j \in \tilde{A}^+$ for every $j$, and by the same argument, $\overline{v}_j \in \tilde{A}^-$ for every $j$, so that $\tilde{A}^+ \cap \tilde{A}^- \neq \emptyset$.

**Corollary 12.** *Every eigenform is a multiple of* $\overline{E}$.

*Proof.* It suffices to observe, that if $E \in \widetilde{\mathcal{D}}$ is an eigenform, then $E = E_{(n)} \underset{n\to\infty}{\longrightarrow} a\overline{E}$, for some $a > 0$. □

Theorem 10 is interesting in itself. However, we will use it as a starting point to prove the $\Gamma$-convergence of $E^{\Sigma}_{(n)}$. Let now $E \in \widetilde{\mathcal{D}}$ and $\tilde{E} = \lim_{n\to\infty} E_{(n)}$. We are going to prove that $\tilde{E}^{\Sigma}_{(\infty)} = \Gamma(X-) \lim_{n\to+\infty} E^{\Sigma}_{(n)}$, where $X = \mathbb{R}^K$ with the metric $L^\infty$. Note that, since $\tilde{E}$ is an eigenform, $\tilde{E}^{\Sigma}_{(\infty)}$ is defined. The argument of the proof of $\Gamma$-convergence is due to S. Kozlov [7], who proved the $\Gamma$-convergence on the Gasket for forms $E$ having two of the three coefficients equal, with respect to a topology which is different from $L^\infty$, and induces a sort of Sobolev space on the Gasket. In any case the proof of [7] also works for general fractals once we know that the sequence $E_{(n)}$ is convergent. In these Notes, I follow an approach similar to that in [17], where, in fact, the problem is treated by a slightly more general point of view in the sense that the $\Gamma$-convergence with respect to different topologies is investigated there.

Recall that, given a sequence of functionals $F_n$ from a metric space $X$ with values in $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, $F_n$ are said to be $\Gamma(X-)$-*convergent* to a functional $F$ if for every $x \in X$

i) there exist $x_n \underset{n\to+\infty}{\longrightarrow} x$ such that $F_n(x_n) \underset{n\to+\infty}{\longrightarrow} F(x)$.

ii) For every $x_n \underset{n\to+\infty}{\longrightarrow} x$ we have $\liminf\limits_{n\to+\infty} F_n(x_n) \geq F(x)$.

In order to obtain the $\Gamma$-convergence result, for every $m, n \in \mathbb{N}$ with $n \geq m$ and for every $v : V^{(m)} \to \mathbb{R}$, we define $v_{(n,m)} : K \to \mathbb{R}$ to be the $n$-harmonic extension with respect to $\tilde{E}$ (or to $\overline{E}$, which is the same as $\tilde{E}$ is a multiple of $\overline{E}$) of $\tilde{v} : V^{(n)} \to \mathbb{R}$ defined by

$$\tilde{v}\big(\psi_{i_1,\dots,i_n}(P)\big) = H_{(n-m;E)}(v \circ \psi_{i_1,\dots,i_m}) \circ \psi_{i_{m+1},\dots,i_n}(P), \quad P \in V^{(0)}\,.$$

The definition of $\tilde{v}$ is correct by the same argument as in Section 2, via Lemma 3. Then, we define $v_{(n)} = v_{(n,[\frac{n}{2}])}$. We have

**Lemma 23.** *For every $v \in C(K)$, $v_{(n)} \underset{n\to\infty}{\longrightarrow} v$ with respect to $L^\infty$.*

*Proof.* For every $Q \in K$ let $i_1, ..., i_n = 1, ..., k$, $P \in K$ be such that $Q = \psi_{i_1,\dots,i_n}(P)$. Then, by the definition of harmonic extension,

$$v_{(n)}(Q) = v_{(n)}(\psi_{i_1,\dots,i_n}(P)) = H_{(\infty;\tilde{E})}\Big(\tilde{v} \circ \psi_{i_1,\dots,i_n}\Big)(P)$$

and by Lemma 5 we have

$$v_{(n)}(Q) \in [\min_{V^{(0)}} \tilde{v} \circ \psi_{i_1,\dots,i_n}, \max_{V^{(0)}} \tilde{v} \circ \psi_{i_1,\dots,i_n}]\,.$$

Moreover, by Corollary 4 we get

$$v_{(n)}(Q) \in \Big[\min_{V^{(0)}} v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}}, \max_{V^{(0)}} v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}}\Big]\,.$$

In conclusion,

$$v_{(n)}(Q), v(Q) \in \big[\min_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v, \max_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v\big]\,,$$

hence $|v_{(n)}(Q) - v(Q)| \leq \max\limits_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v - \min\limits_{K_{i_1,\dots,i_{[\frac{n}{2}]}}} v$, and by the uniform continuity of $v$ on $K$ and (11), $v_{(n)} \underset{n\to\infty}{\longrightarrow} v$ uniformly. □

**Lemma 24.** *For every $v \in C(K)$ we have*

$$E^{\Sigma}_{(n)}(v_{(n)}) = (E_{(n-[\frac{n}{2}])})^{\Sigma}_{([\frac{n}{2}])}(v) \leq E^{\Sigma}_{(n)}(v)\,.$$

*Proof.* We first observe that

$$S_n(E)(v_{(n)}) = \sum_{i_1,..,i_{[\frac{n}{2}]}=1}^{k} S_{n-[\frac{n}{2}]}(E)(v_{(n)} \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}})\,,$$

$$S_{n-[\frac{n}{2}]}(E)(v_{(n)} \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}}) = S_{n-[\frac{n}{2}]}(E)\big(H_{(n-[\frac{n}{2}];E)}(v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}})\big) =$$

$$M_{n-[\frac{n}{2}]}(E)(v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}}) \leq S_{n-[\frac{n}{2}]}(E)(v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}})\,,$$

$$S_n(E)(v) = \sum_{i_1,..,i_{[\frac{n}{2}]}=1}^{k} S_{n-[\frac{n}{2}]}(E)(v \circ \psi_{i_1,\dots,i_{[\frac{n}{2}]}})\,.$$

It follows that

$$S_n(E)(v_{(n)}) = S_{[\frac{n}{2}]}\big(M_{n-[\frac{n}{2}]}(E)\big)(v) \leq S_n(E)(v)\,.$$

By dividing by $\rho^n$ we get the Lemma. □

**Theorem 11.** *We have*

$$\Gamma(X-)\lim_{n\to+\infty} E^{\Sigma}_{(n)} = \tilde{E}^{\Sigma}_{(\infty)}$$

*where $X = C(K)$ with the metric $L^\infty$.*

*Proof.* By Corollary 6, for every $\varepsilon > 0$ we have

$$(1-\varepsilon)\tilde{E} \leq E_{(n)} \leq (1+\varepsilon)\tilde{E} \tag{25}$$

for sufficiently large $n$. Now, given $v \in X$, by Lemma 23 $v_{(n)} \underset{n\to\infty}{\longrightarrow} v$ in $X$. Also, by Lemma 24 and (25),

$$E^{\Sigma}_{(n)}(v_{(n)}) = (E_{(n-[\frac{n}{2}])})^{\Sigma}_{([\frac{n}{2}])}(v) \in \big[(1-\varepsilon)\tilde{E}^{\Sigma}_{([\frac{n}{2}])}(v), (1+\varepsilon)\tilde{E}^{\Sigma}_{([\frac{n}{2}])}(v)\big]$$

for sufficiently large $n$, so that $E^{\Sigma}_{(n)}(v_{(n)}) \underset{n\to\infty}{\longrightarrow} \tilde{E}^{\Sigma}_{(\infty)}(v)$. It remains to prove that given $v_n \underset{n\to+\infty}{\longrightarrow} v$ in $X$, then

$$\liminf_{n\to+\infty} E^{\Sigma}_{(n)}(v_n) \geq \tilde{E}^{\Sigma}_{(\infty)}(v)\,,$$

and it is clearly sufficient to show that for every $m \in \mathbb{N}$

$$\liminf_{n\to+\infty} E^{\Sigma}_{(n)}(v_n) \geq \tilde{E}^{\Sigma}_{(m)}(v)\,. \tag{26}$$

By Lemma 24 and (25) and since $\tilde{E}^{\Sigma}_{(n)}$ is increasing, if $0 < \varepsilon < 1$, for sufficiently large $n$ we have

$$E^{\Sigma}_{(n)}(v_n) \geq (E_{(n-[\frac{n}{2}])})^{\Sigma}_{([\frac{n}{2}])}(v_n) \geq (1-\varepsilon)\tilde{E}^{\Sigma}_{([\frac{n}{2}])}(v_n) \geq (1-\varepsilon)\tilde{E}^{\Sigma}_{(m)}(v_n)\,.$$

Since $v_n \underset{n\to+\infty}{\longrightarrow} v$ and $\tilde{E}^{\Sigma}_{(m)}$ is continuous from $X$ to $\mathbb{R}$, we get

$$\liminf_{n\to+\infty} E^{\Sigma}_{(n)}(v_n) \geq (1-\varepsilon)\liminf_{n\to+\infty} \tilde{E}^{\Sigma}_{(m)}(v_n) = (1-\varepsilon)\tilde{E}^{\Sigma}_{(m)}(v)\,.$$

Since this holds for any $\varepsilon \in ]0,1[$, (26) follows. □

## 6 Homogenization on general fractals

In this section we will extend the results of Section 5 to arbitrary finitely ramified fractals. The difficulty in imitating the proof in Section 5 consists in extending Theorem 10. Note however, that in general, we cannot expect that $E_{(n)}$ tends to a multiple of a fixed eigenform, as in such a case, we could prove as in Corollary 12, that the eigenform is unique up to a multiplicative constant, and this is no longer true in the general case. We remark that if $N = 2$, then for every $E \in \widetilde{\mathcal{D}}$, we have

$$E(u) = c\big(u(P_1) - u(P_2)\big)^2, \qquad M_1(E)(u) = c'\big(u(P_1) - u(P_2)\big)^2$$

for some $c, c' > 0$. Hence, $E$ is in any case an eigenform, and the convergence of $E_{(n)}$ takes trivially place. Thus, we can assume $N \geq 3$. What properties of the Gasket have we used in the proof of the convergence of $E_{(n)}$? We essentially used either $N = 3$ or the very strong symmetry of the Gasket. If the fractal is less symmetric, one could prove an analog of Lemma 22, but the limit vector is not symmetric with respect to the components different from $j$. Hence such a vector does not attain its maximum at *all* $P \in V^{(0)}$, $P \neq P_j$, and the proof in Remark 7 does not work. Since in the present case, we want to prove the convergence of $E_{(n)}$ to an eigenform that, in case of nonuniqueness, may well depend on $E$, and we cannot guess what eigenform is the limit, we will not try to prove that $\lambda(\tilde{E}, E_{(n)})$ tends to 0 for some specific eigenform $\tilde{E}$. We will instead try to prove that $\lambda_n := \lambda(E_{(n)}, E_{(n+1)})$ tends to 0. Note that $E_{(n+1)} = (E_{(1)})_{(n)}$, so that we can use Corollary 8 and Corollary 10 with $E_{(1)}$ in place of $E'$. In particular, $\lambda_n$ is decreasing. We say that $E$ is *$\lambda$-contracting* if we have $\lambda_n < \lambda_0$ for some, thus for sufficiently large, $n$. We have the following analog to Lemma 21.

**Lemma 25.** *If every $E \in \widetilde{\mathcal{D}}$ that is not an eigenform is $\lambda$-contracting, then for every $E \in \widetilde{\mathcal{D}}$ there exists an eigenform $\tilde{E}$ such that $E_{(n)} \underset{n\to\infty}{\longrightarrow} \tilde{E}$.*

*Proof.* Let $E \in \widetilde{\mathcal{D}}$. By Corollary 7 there exists a strictly increasing sequence of naturals $n_h$, and $E' \in \widetilde{\mathcal{D}}$ such that $E_{(n_h)} \underset{h\to\infty}{\longrightarrow} E'$. For every $m \in \mathbb{N}$ we have

$$\lambda(E'_{(m)}, E'_{(m+1)}) = \lambda\big(\lim_{h\to\infty}(E_{(n_h)})_{(m)}, \lim_{h\to\infty}(E_{(n_h)})_{(m+1)}\big) =$$

$$\lambda(\lim_{h\to\infty} E_{(m+n_h)}, \lim_{h\to\infty} E_{(m+1+n_h)}) = \lim_{h\to\infty} \lambda(E_{(m+n_h)}, E_{(m+1+n_h)})$$
$$= \lim_{n\to\infty} \lambda(E_{(n)}, E_{(n+1)}) .$$

Thus $E'$ is not $\lambda$-contracting, and by hypothesis it is an eigenform. By Corollary 9, then $E_{(n)} \underset{n\to\infty}{\longrightarrow} E'$. □

Actually, in the argument of the proof of convergence on the Gasket, when we stated that $a$ and $b$ in (23) are positive, we also used the strong maximum principle. The strong maximum principle, in fact, will play an important role also in the argument in this section. This will lead us to restrict the class of fractals. The convergence result can be proved for all finitely ramified fractals, using a variant of the strong maximum principle, but the proof is considerably more technical and will be omitted. I will hint the idea at the end of this section. We saw in Prop. 1 and Remark 4 that $H_{(1;E)}(u)$ satisfies the maximum principle, but in general not the strong maximum principle when $E \in \widetilde{\mathcal{D}}$. We will now see that, if

$$c_{j_1,j_2}(E) > 0 \quad \forall\, j_1, j_2 : j_1 \neq j_2 \,, \tag{27}$$

then $H_{(1;E)}(u)$ even satisfies the strong maximum principle. Note that, if $E \in \widetilde{\mathcal{D}}$ satisfies (27), then $E$-close amounts to close and $E$-connected amounts to connected.

**Proposition 5.** *Suppose that $E \in \widetilde{\mathcal{D}}$ and (27) holds. Suppose $u : V^{(0)} \to \mathbb{R}$, and $v := H_{(1;E)}(u)$ attains its maximum or its minimum at a point $Q$ of $V^{(1)} \setminus V^{(0)}$. Then $u$ is constant on $V^{(0)}$.*

*Proof.* We first prove that any point $Q \in V^{(1)} \setminus V^{(0)}$ is strongly $E$-connected to any point $Q' \in V^{(0)}$. The proof of Lemma 7 shows that there exists a path $(Q_1, ..., Q_m)$ $E$-connecting $Q$ to $Q'$. We will prove that if such a path has minimum length among the paths $E$-connecting $Q$ and $Q'$, then $Q_2, ..., Q_{m-1} \notin V^{(0)}$. Suppose on the contrary $Q_{\bar{i}} = P_h \in V^{(0)}$ with $1 < \bar{i} < m$. As $V_h$ is the unique 1-cell containing $P_h$, then $Q_{\bar{i}-1}, Q_{\bar{i}+1} \in V_h$. Hence the path $(Q_1, ..., Q_{\bar{i}-1}, Q_{\bar{i}+1}, ..., Q_m)$ $E$-connects $Q$ and $Q'$ and has length $m-1$, which contradicts our assumption. Now, from Lemma 9, if $v$ attains its maximum or its minimum at $Q$, then $u = v = v(Q)$ on $V^{(0)}$. □

We cannot apply directly Prop. 5 to prove the convergence of $E_{(n)}$, as $E$ does not necessarily satisfy (27), but we will now see that a relatively mild condition on the fractal implies that every $M_1(E)$ satisfies (27). We first give a sufficient condition on $E$ in order that $M_1(E)$ satisfy (27). It can be easily proved that it is also a necessary condition.

**Lemma 26.** *Suppose that $E \in \widetilde{\mathcal{D}}$ and every two points in $V^{(0)}$ are strongly $E$-connected. Then $M_1(E)$ satisfies (27).*

*Proof.* The proof is a variant of that of b) in Theorem 7. Let $j_1, j_2, u_{j_1,j_2}, v_{j_1,j_2}$, $w$ be as in Theorem 7. It suffices to prove that $M_1(E)(v_{j_1,j_2}) > M_1(E)(u_{j_1,j_2})$, hence that in (17) at least one of the two inequalities is strict. If the second one is not strict, we have $E(|w| \circ \psi_i) = E(w \circ \psi_i)$ for all $i = 1, ..., k$, and thus $w$ cannot attain opposite signs at two $E$-close points, as the inequality in (18) is strict when $a$ and $b$ have opposite signs. Let now $(Q_1, ..., Q_m)$ be a path strongly $E$-connecting $P_{j_1}$ to $P_{j_2}$. Here $m > 2$ as $P_{j_1}$ and $P_{j_2}$ cannot lie in the same 1-cell. Since $w(Q_1) = w(P_{j_1}) = v_{j_1,j_2}(P_{j_1}) = 1$ and $w(Q_m) = w(P_{j_2}) = v_{j_1,j_2}(P_{j_2}) = -1$, there exists $h = 2, ..., m-1$ such that $w(Q_h) = 0$. As $|w|$ does not satisfy Lemma 9 with $P = Q_h$, then $|w| \neq H_{(1;E)}(u_{j_1,j_2})$, and the first inequality in (17) is strict. □

We now require that the fractal has a strong connectedness property. The argument in proof of Prop. 5 shows that any two points in $V^{(0)}$ are strongly connected. We require a slightly stronger condition. We say that the fractal is *strongly connected* if for every $Q, Q' \in V^{(0)}$ there exist $Q_1, ..., Q_m \in V^{(1)}$ such that $Q_1 = Q$, $Q_m = Q'$ and, for every $h = 1, ..., m-1$ there exists $i(h) = 1, ..., k$ such that $Q_h, Q_{h+1} \in V_{i(h)}$, and in addition, $i(h) > N$ for $h = 2, ..., m-2$. Note that the last condition means that all cells but the first and the last contain no points of $V^{(0)}$. It is easy to see that the Gasket, the Vicsek set and the Snowflake are strongly connected, while the tree-like Gasket is not.

**Proposition 6.** *Suppose the fractal is strongly connected and $E \in M_1(\widetilde{\mathcal{D}})$. Then $E$ satisfies (27). Thus $H_{(1;E)}(u)$ satisfies the strong maximum principle for every $u \in \mathbb{R}^{V^{(0)}}$.*

*Proof.* Let $E = M_1(E')$, $E' \in \widetilde{\mathcal{D}}$. We will prove that every two points in $V^{(0)}$ are strongly $E'$-connected. The proof imitates that of Lemma 7. Fix $P_{j_1}, P_{j_2} \in V^{(0)}$. Let $(Q_1, ..., Q_m)$ be a path as in definition of strongly connected fractal. For any $h = 1, ..., m-1$, the points $Q_h$ and $Q_{h+1}$ are $E'$-connected by a path which remains in $V_{i(h)}$, in particular, for $h = 2, ..., m-2$, it remains in $V^{(1)} \setminus V^{(0)}$. Since $Q_2 \in V_{j_1}$ (the unique 1-cell containing $P_{j_1}$), by Remark 3 $Q_2$ is strongly $E'$-connected to $P_{j_1}$. For the same reason, $Q_{m-1}$ is strongly $E'$-connected to $P_{j_2}$. In conclusion, we have found a path that strongly $E'$-connects $P_{j_1}$ and $P_{j_2}$. □

*From now on, unless specified otherwise, we will assume that the fractal is strongly connected.*

We will write any $u \in \mathbb{R}^{V^{(0)}}$ as $(u(P_1), ..., u(P_N))$. In this way, $u$ will be identified to a vector in $\mathbb{R}^N$. We will denote by $e_1, ..., e_N$ the vectors of the canonical basis in $\mathbb{R}^N$.

**Corollary 13.** *Suppose $E \in M_1(\widetilde{\mathcal{D}})$. Then for every $i = 1, ..., k$, $j = 1, ..., N$ we have*

$$T_{i;E}(e_h)(P_j) \begin{cases} = 1 & \text{if } i = j = h \\ = 0 & \text{if } i = j \neq h \\ \in ]0,1[ & \text{if } i \neq j \, . \end{cases}$$

*Proof.* We have $T_{i;E}(e_h)(P_j) = H_{(1;E)}(e_h)(\psi_i(P_j))$. If $i \neq j$ we have $\psi_i(P_j) \in V^{(1)} \setminus V^{(0)}$, thus the conclusion follows from the strong maximum principle. If $i = j$ the results is trivial. □

In the previous section, in order to prove the convergence of $E_{(n)}$, we were lead, in view of Lemma 21, to investigate the implications of the equality $\lambda_\pm(\overline{E}, E_{(n)}) = \lambda_\pm(\overline{E}, E)$. In the present case, in order to prove the convergence of $E_{(n)}$, in view of Lemma 25, we have to investigate the implications of the equality $\lambda(E, E_{(1)}) = \lambda(E_{(n)}, E_{(n+1)})$. Therefore, instead of using a version of Corollary 10 in the case in which $E$ is an eigenform, we will use a version of Corollary 10 in the case $E' = E_{(1)}$. However, also in the present case, it will be sufficient to consider the case $i_1 = ... = i_m$. We have

**Lemma 27.** *Suppose $E \in \widetilde{\mathcal{D}}$. Let $\lambda_{\pm,n} = \lambda_\pm(E_{(n)}, E_{(n+1)})$ for all $n$. If we have $\lambda_{\pm,n} = \lambda_{\pm,0}$, then for all $m$ with $0 \leq m \leq n$ we have $\lambda_{\pm,m} = \lambda_{\pm,0}$ and if $u \in A^{\pm,n}(E, E_{(1)})$*

$$T_{i,m} \circ \cdots \circ T_{i,n-1}(u) = T_{i,m+1} \circ \cdots \circ T_{i,n}(u) \in A^{\pm,m}(E, E_{(1)}) \, .$$

*Proof.* It suffices to put $E' = E_{(1)}$, $i_1 = \cdots = i_m = i$ in Corollary 10, and to use Lemma 15, taking into account the definition of $H_{(n-m,m;E)}$. □

We need an analog of Lemma 22 suitable for Lemma 27. Roughly speaking, we want to prove that the normalization of the composition of $m$ linear operators, under suitable conditions, contracts the positive cone to a unique vector, which in this case may well be nonsymmetric. However, there is a more relevant difference with the case of Section 5, i.e., as suggested by Lemma 27 we need to consider the case where the operators are different. I will give a general theorem for operators in $\mathbb{R}^M$ and then I will apply this to our case. Let $M \geq 2$ be fixed for the following (in case $M = 1$, Theorem 12 is trivially satisfied with $w = 1$). We define

$$D = \left\{ v \in \mathbb{R}^M : v_i \geq 0 \; \forall \, i = 1, ..., M, v \neq 0 \right\},$$

$$D' = \left\{ v \in \mathbb{R}^M : v_i > 0 \ \forall\, i = 1, ..., M \right\}.$$

Let also $\tilde{D} = \{v \in D : ||v|| = 1\}$, $\tilde{D}' = \{v \in D' : ||v|| = 1\}$. Let $\mathcal{A}$ be the set of linear operators $T : \mathbb{R}^M \to \mathbb{R}^M$ that map $D$ into $D'$ or equivalently such that $T(e_i) \in D'$ for all $i = 1, ..., M$. Let $\mathcal{N} : \mathbb{R}^M \setminus \{0\} \to \mathbb{R}^M$ be defined as $\mathcal{N}(v) = \frac{v}{||v||}$. We will now introduce on $D'$ a semimetric that we implicitly used in the proof of Lemma 22 with $M = 2$, and is standard in the Perron-Frobenius Theory. Given $u, v \in D'$ put

$$\lambda'_+(u, v) = \max_{i=1,\dots,M} \frac{v_i}{u_i}, \quad \lambda'_-(u, v) = \min_{i=1,\dots,M} \frac{v_i}{u_i},$$
$$\lambda'(u, v) = \ln(\lambda'_+(u, v)) - \ln(\lambda'_-(u, v)).$$

The definition of $\lambda'$ resembles that of $\lambda$ on $\widetilde{\mathcal{D}}$, and in fact it is another case of Hilbert's projective metric. It satisfies the same properties as $\lambda$: we have $\lambda'(u, v) = 0$ if and only if $v$ is a multiple of $u$, $\lambda'(au, bv) = \lambda'(u, v)$ for every $a, b > 0$. Thus, $\lambda'$ induces a metric on the projective space $pr(D')$ generated by $D'$, that is the space of equivalence classes on $D'$, mod the relation that identifies $v$ to $av$ for $a > 0$, $v \in D'$. In particular, on $\tilde{D}'$, it induces a metric, which we denote by $\lambda'$ as well. It is not difficult to see that such a metric is equivalent to the euclidean one, in the sense that they generate the same topology. However, as we will not use that statement, I will not discuss it. We instead will use a weaker form in Theorem 12. We will denote the diameter of a subset $A$ of $\tilde{D}'$ with respect to $\lambda'$ by $\mathrm{diam} A$. The use of $\lambda'$ is illustrated by the following lemma.

**Lemma 28.** *If $u, v \in D'$ and $T \in \mathcal{A}$, then we have $\lambda'(T(u), T(v)) \leq \lambda'(u, v)$ and the inequality is strict unless $\lambda'(u, v) = 0$.*

*Proof.* For all $i = 1, ..., M$, we have

$$\frac{T(v)_i}{T(u)_i} = \frac{\sum_{j=1}^{M} a_{i,j} v_j}{\sum_{j=1}^{M} a_{i,j} u_j}$$

where $a_{i,j} = \big(T(e_j)\big)_i > 0$. We have $a_{i,j} v_j \leq \lambda'_+(u, v) a_{i,j} u_j$ and, if $\lambda'(u, v) \neq 0$, i.e., $v$ is not a multiple of $u$, the inequality is strict for at least one $j$. Thus $\frac{T(v)_i}{T(u)_i} \leq \lambda'_+(u, v)$ for every $i = 1, ..., M$. Hence, $\lambda'_+(T(u), T(v)) \leq \lambda'_+(u, v)$ and similarly, $\lambda'_-(T(u), T(v)) \geq \lambda'_-(u, v)$, thus $\lambda'(T(u), T(v)) \leq \lambda'(u, v)$, and the inequality is strict if $\lambda'(u, v) \neq 0$. □

The following theorem is specially interesting in the case in which the operators coincide. This is a form of the well-known Perron-Frobenius Theorem. For information on the Perron-Frobenius theory see for example [19].

**Theorem 12.** *Let $\mathcal{A}'$ be a compact subset of $\mathcal{A}$. Let $T_1, ..., T_n, ...$ be a sequence of operators in $\mathcal{A}'$. Then there exists $w \in \tilde{D}'$ such that*

$$\mathcal{N}\big(T_1 \circ ... \circ T_n(D)\big) \underset{n\to\infty}{\longrightarrow} w$$

*in the sense that* $\sup\Big\{||w - v|| : v \in \mathcal{N}\big(T_1 \circ ... \circ T_n(D)\big)\Big\} \underset{n\to\infty}{\longrightarrow} 0.$

*Proof.* For every $T \in \mathcal{A}$ let $\tilde{T} = \mathcal{N} \circ T$. Then, we have

$$\mathcal{N}\big(T_1 \circ ... \circ T_n(D)\big) = \tilde{T}_1 \circ ... \circ \tilde{T}_n(\tilde{D}) \quad \forall\, n \in \mathbb{N}\,.$$

Thus, putting $\tilde{T}_{m,n} = \tilde{T}_m \circ ... \circ \tilde{T}_n$ when $m \le n$, we have to prove that there exists $w \in \tilde{D}'$ such that

$$\sup\big\{||w - v|| : v \in \tilde{T}_{1,n}(\tilde{D})\big\} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{28}$$

Let $\alpha : \tilde{D} \times \mathcal{A}' \to \tilde{D}'$ be defined as $\alpha(v, T) = \tilde{T}(v)$. It is continuous, hence $B :=$Im$\alpha$ is a compact subset of $\tilde{D}'$, thus, $\overline{M} :=$diam$(B) < +\infty$. We have $\tilde{T}_n(\tilde{D}) \subseteq B \subseteq \tilde{D}$. It follows

$$B \supseteq \tilde{T}_{1,1}(\tilde{D}) \supseteq \tilde{T}_{1,2}(\tilde{D}) \supseteq ... \supseteq \tilde{T}_{1,n}(\tilde{D}) \supseteq ... \tag{29}$$

We will prove that

$$\text{diam}(\tilde{T}_{1,n}(\tilde{D})) \underset{n\to\infty}{\longrightarrow} 0\,. \tag{30}$$

For every $\eta > 0$ let $F_\eta = \{(u, v) \in B \times B : \lambda'(u, v) \ge \eta\}$ and let $\beta : \mathcal{A}' \times F_\eta \to \mathbb{R}$ be defined as

$$\beta(T, u, v) = \lambda'(u, v) - \lambda'\big(\tilde{T}(u), \tilde{T}(v)\big)\,.$$

Since $\mathcal{A}' \times F_\eta$ is compact, $\beta$ has a minimum $m_\eta$ on it which, by Lemma 28, is positive. We will now prove that, given $n = 1, 2, 3, ...$ such that $M - n m_\eta < \eta$, we have

$$\text{diam}\tilde{T}_{1,n+1}(\tilde{D}) \le \eta\,. \tag{31}$$

As $\eta$ is arbitrary, this implies (30). Let $u, v \in \tilde{D}$. As $\tilde{T}_{n+1}(u), \tilde{T}_{n+1}(v) \in B$, we have

$$\lambda'(\tilde{T}_{n+1}(u), \tilde{T}_{n+1}(v)) \le M\,.$$

Now, if $\lambda'(\tilde{T}_{m,n+1}(u), \tilde{T}_{m,n+1}(v)) < \eta$ for some $m \le n + 1$, by Lemma 28 we have $\lambda'(\tilde{T}_{1,n+1}(u), \tilde{T}_{1,n+1}(v)) < \eta$. In the contrary case, we have $\big(\tilde{T}_{m,n+1}(u), \tilde{T}_{m,n+1}(v)\big) \in F_\eta$ for all $m \le n+1$. Hence, in view of the definition of $m_\eta$, by a recursive argument we get $\lambda'(\tilde{T}_{1,n+1}(u), \tilde{T}_{1,n+1}(v)) \le M - n m_\eta < \eta$, and $\lambda'(\tilde{T}_{1,n+1}(u), \tilde{T}_{1,n+1}(v)) < \eta$ again. Thus, we have proved (31), hence also (30). Since by (29) $\tilde{T}_{1,n}(\tilde{D})$ is a decreasing sequence of nonempty compact subsets of $B$, then there exists $w \in \bigcap_{n=1}^{\infty} \tilde{T}_{1,n}(\tilde{D})$, and, by (30) we have

$$\sup\big\{\lambda'(w,v) : v \in \tilde{T}_{1,n}(\tilde{D})\big\} \underset{n\to\infty}{\longrightarrow} 0\,.$$

On the other hand, since $\lambda'$ generates on the euclidean compact set $B$ a topology weaker than the euclidean one, $\lambda'$ is equivalent to the euclidean metric. Hence, we get (28), and the theorem is proved. □

**Corollary 14.** *Under the hypotheses of Theorem 12, given a strictly increasing sequence $n_h$ of naturals and $v_h \in D$ we have*

$$\mathcal{N}\big(T_1 \circ ... \circ T_{n_h}(v_h)\big) \underset{h\to\infty}{\longrightarrow} w\,.$$

**Corollary 15.** *Let $i = 1, ..., k$ be fixed, let $E \in \widetilde{\mathcal{D}}$ and let $D = \big\{v \in \mathbb{R}^{V^{(0)}} : v(P_i) = 0, v(P_{i'}) \geq 0 \ \ \forall\, i' \neq i, v \neq 0\big\}$. Then for every $m = 1, 2, ...$, there exists $w_m \in \mathbb{R}^{V^{(0)}}$ such that, for every strictly increasing sequence $n_h$ of naturals and for every $v_h \in D$, we have*

$$\mathcal{N}\big(T_{i,m} \circ \cdots \circ T_{i,n_h}(v_h)\big) \underset{h\to\infty}{\longrightarrow} w_m\,.$$

*Proof.* As previously seen, the operators $T_{i,n}$ can be identified to linear operators from $\mathbb{R}^{N-1}$ into itself, as they map the $(N-1)$-dimensional linear space $\Pi_i$ into itself. By this identification, $D$ corresponds to the previously defined $D$, with $N-1$ in place of $M$. Given $a, b > 0$ such that $\frac{E}{\rho} \in U_{a,b}$, for all $m \geq 1$ we have $E_{(m)} = M_1\big((\frac{E}{\rho})_{(m-1)}\big) \in M_1(U_{a,b})$, hence $T_{i,m} = T_{i;E_{(m)}} \in T_{i;M_1(U_{a,b})}$. By Lemma 19 and Lemma 17, the set $T_{i;M_1(U_{a,b})}$ is compact, and by Corollary 13 it is contained in $\mathcal{A}$. We can now use Corollary 14. □

We are now ready to prove the main theorem in this section.

**Theorem 13.** *For every $E \in \widetilde{\mathcal{D}}$ there exists an eigenform $\tilde{E} \in \widetilde{\mathcal{D}}$ such that $E_{(n)} \underset{n\to\infty}{\longrightarrow} \tilde{E}$.*

*Proof.* By Lemma 25 it suffices to prove that, given any non $\lambda$-contracting $E \in \widetilde{\mathcal{D}}$, then $E$ is an eigenform. For every natural $n$ let $u_n \in \tilde{A}^{+,n}(E, E_{(1)})$. Let $n_h$ be a strictly increasing sequence of naturals and let $i = 1, ..., N$ be such that $\min u_{n_h} = u_{n_h}(P_i)$ for all $h \in \mathbb{N}$. We can assume that $u_{n_h}(P_i) = 0$, so that $u_{n_h} \in D$ where $D$ is defined as in Corollary 15. Since $E$ is not $\lambda$-contracting, by Remark 6, $\lambda_\pm(E_{(n)}, E_{(n+1)}) = \lambda_\pm(E, E_{(1)})$ and we can apply Lemma 27, hence

$$u_{m,n_h} := \mathcal{N}\big(T_{i,m,n_h-1}(u_{n_h})\big) = \mathcal{N}\big(T_{i,m+1,n_h}(u_{n_h})\big) \in \tilde{A}^{+,m}(E, E_{(1)}) \quad (32)$$

for all $m \geq 1$, where $T_{i,m,n}$ denotes $T_{i,m} \circ \cdots \circ T_{i,n}$ when $m \leq n$. Thus, using Corollary 15, there exists $w_m \in \mathbb{R}^{V^{(0)}}$ such that $u_{m,n_h} \to w_m = w_{m+1}$ as $h \to +\infty$. Hence, there exists a unit vector $w \in \mathbb{R}^{V^{(0)}}$ such that $w_m = w$ for all $m \geq 1$. It follows from (32) that $w \in \tilde{A}^{+,m}(E, E_{(1)})$ for all $m \geq 1$, so that $E_{(m+1)}(w) = \lambda_+ E_{(m)}(w)$, where we put $\lambda_\pm = \lambda_\pm(E, E_{(1)})$, hence

$$E_{(m)}(w) = \lambda_+^{m-1} E_{(1)}(w),$$

for all $m \geq 1$. By the same argument, there exists $w' \in \tilde{A}^{-,m}(E, E_{(1)})$ such that

$$E_{(m)}(w') = \lambda_-^{m-1} E_{(1)}(w')$$

for all $m \geq 1$. But we know that for some $a, b > 0$ we have $E_{(m)} \in U_{a,b}$ for all naturals $m$. Thus,

$$E_{(m)}(w) \leq b\hat{E}(w) = b\hat{E}(w') \frac{\hat{E}(w)}{\hat{E}(w')} \leq L E_{(m)}(w')\,,$$

where $L = \frac{b}{a} \frac{\hat{E}(w)}{\hat{E}(w')}$. This is possible only if $\lambda_- = \lambda_+$, i.e., only if $E$ is an eigenform. □

We can now prove a $\Gamma$-convergence result in the exactly same way as in Section 5. Namely,

**Theorem 14.** *Let $E \in \widetilde{\mathcal{D}}$. We have*

$$\Gamma(X-) \lim_{n \to +\infty} E^{\Sigma}_{(n)} = \tilde{E}^{\Sigma}_{(\infty)}$$

*where $\tilde{E} = \lim_{n \to \infty} E_{(n)}$ and $X = C(K)$ with the metric $L^\infty$.* □

As we previously saw, Theorem 13 holds, even if the fractal is not strongly connected. I now sketch the idea of the proof in this case. The problem is that, as previously hinted, the strong maximum principle does not hold. So, we cannot use Theorem 12 to deduce Corollary 15. However, we can prove the following variant of Theorem 12. For every $B \subseteq \{1, ..., N\}$, let

$$\Pi_B = \left\{ v \in \mathbb{R}^N : v_i = 0 \quad \forall i \notin B \right\},$$

$$D_B = \left\{ v \in \Pi_B : v_i \geq 0 \quad \forall i \in B, v \neq 0 \right\},$$

$$D'_B = \left\{ v \in \Pi_B : v_i > 0 \quad \forall i \in B \right\}.$$

**Theorem 15.** *Suppose $T_1, .., T_n, ..., T_\infty$ are linear maps from $\mathbb{R}^N$ to $\mathbb{R}^N$, and there exists $B \subseteq \{1, ..., N\}$ such that*

*i) $T_n$ maps $\Pi_B$ into itself for every $n \in \mathbb{N} \cup \{\infty\}$*

*ii) $T_n$ maps $D_B$ into $D'_B \cup \{0\}$ for every $n \in \mathbb{N} \cup \{\infty\}$*

*iii) There exists a strictly increasing sequence of indices $n_h$ such that $T_{n_h} \underset{h\to\infty}{\longrightarrow} T_\infty$, and $D_B \cap Ker T_{n_h} = D_B \cap Ker T_\infty$ for every $h \in \mathbb{N}$.*

*Then there exists $\overline{v} \in D'$ such that, for every strictly increasing sequence of naturals $n_h$, for every $v_h \in D'$ we have $\mathcal{N}(T_1 \circ ... \circ T_{n_h}(v_h)) \underset{h\to\infty}{\longrightarrow} \overline{v}$ .*

In other words, the condition that $T_n$ maps $D$ into $D'$ is replaced by the condition that $T_n$ maps $D$ into $D' \cup \{0\}$, and the part of $D$ mapped into 0 is constant on a suitable compact set containing a subsequence. Theorem 15 fits in the situation of fractals which are not strongly connected. In fact, the positivity of $T_{i,n;E}(e_j)(P_{j'})$ is related to the graph $\mathcal{G}(E_{(n)})$. Here we identify $V^{(0)}$ with $\{1, ..., N\}$, and accordingly we interpret $\Pi_B$, $D_B$, $D'_B$ . Usually, $\mathcal{G}(E_{(n)})$ can change at any step, and it need not satisfy the hypotheses of Theorem 15 if we do not require some additional hypothesis. However, this is the case if $E$ is not $\lambda$-contracting. More precisely, we have:

**Lemma 29.** *There exists $n_3 \geq 1$ such that, if $h \geq n_1 + n_2 + n_3$, $E \in \widetilde{\mathcal{D}}$, $u \in \tilde{A}^{\pm,h}(E, E_{(1)})$ with $\lambda_{\pm,h}(E, E_{(1)}) = \lambda_\pm(E, E_{(1)})$, then there exist $m$ with $0 \leq m \leq n_3$, $i_1, ..., i_m = 1, ..., k$, $j = 1, ..., N$, $B \subseteq V^{(0)} \setminus \{P_j\}$, $a = 1, -1$, such that*

*i) $a\big(H_{m,h-m}(u) \circ \psi_{i_1,...,i_m} - c\big) \in D'$ where $c = H_{m,h-m}(u) \circ \psi_{i_1,...,i_m}(P_j)$.*

*ii) $T_n := T_{j,n+n_1}$, satisfies i) and ii) of Theorem 15.*

Of course, we here identify $\mathbb{R}^{V^{(0)}}$ with $\mathbb{R}^N$, and, by this identification, $v(P_j)$ corresponds to $v_j$. In other words, while in the case of strong maximum principle we can start with a function $u \in D$, which is mapped into $D'$ by one operator $T_{j,n}$, in the present case i) states that we have to use $m$ operators before mapping $u$ into $D'$. After doing this, on the base of ii), we are able to apply Theorem 15. We have, in fact, still to verify iii) of Theorem 15, which however follows from a compactness argument. In fact, such a compactness argument holds for $E'$ in place of $E$ where $E'$ is a suitable limit point of $E_{(n)}$, namely a limit point for which the graph $\mathcal{G}(E')$ has the minimal number of elements. Lemma 29 plays in this more general case the role that Corollary 13 played in the case of strongly connected fractals. The proof of Theorem 15 is a not too complicated variant of the proof of Theorem 12. On the contrary, the proof of Lemma 29 is the most delicate step in the proof of convergence of $E_{(n)}$. At this moment, I do not know any simplification of that proof. Using Theorem 15 and Lemma 29 it is possible to prove again Theorem 13, and consequently, Theorem 14. In the proof of Theorem 13, however, there are some additional technical points with respect to the case of strongly connected fractals. Complete details can be found in [16], Section 4. There, Theorem 13 was proved in the more general setting of combinatorial fractal structures. The idea of investigating combinatorial fractal structures, firstly considered in [3], is motivated by the fact that $S_n(E)$, $M_n(E)$ and so on only depend on the graph $\mathcal{G}_1(E)$ and not on the geometry of the fractal. Theorem 13 was proved in [16] in its full generality, i.e., fractal structures having an eigenform. For previous proofs in particular cases, see for example [7] and [11], Example 8.8 and references therein for the case of Gasket, [12] for nested fractals with coefficients only depending on the distance; in [14] the result in strongly symmetric fractals (e.g., the Gasket) was announced without proof.

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# Homogenization in perforated domains

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## Introduction

The main goal of these lectures is to show the effect which one can discover studying problems in perforated domains. In different situations the effective behavior of such domains is described by the homogenized equation involving a "term etrange" (additional nontrivial potential). First this effect was studied in [21]. The proofs in [21] are complicated but the authors considered the nonperiodic general case. They used the method of potentials for solving boundary value problems. The notion of "term etrange" was introduced in the famous paper [13], where the authors considered the periodic case and suggested an elegant proof of the homogenization result. Now one can find different papers where other methods were used for studying such problems. See for instance [6], [7], [8] [14], [26], [27], [31].

We want to present two main types of such situations. The first one is connected with the investigation of a boundary value problem in perforated domain with low concentration of holes and Dirichlet boundary conditions on the holes; in the second situation we study a boundary value problem in a perforated domain with Fourier boundary conditions posed on the boundary of the holes. We study the problem in the second case assuming that either the coefficients in the boundary condition on the perforation, are small or the mean values of the coefficients are small (see also [6]).

At present, there is a rich collection of results on asymptotic analysis of problems in perforated domains, among them the homogenization results obtained for periodic, almost-periodic, and random structures. A detailed bibliography can be found, for example, in [11], [17], [19], [20], [21], [25], [30]. In particular, the problems involving Fourier–type boundary conditions at the boundary of the holes, were studied in [5], [7], [8] [10], [16], [27], [28], [31], [32] and [33].

In the special case, when the problems under consideration are dissipative (which is ensured by proper signs of the coefficients in the boundary conditions), the weak convergence of solutions was investigated in [14], [15], [16].

Applying the method of compensated compactness [12], [22] and the two-scale convergence method [1], [24] (see also [23], where the method of two-scale convergence was adapted for perforated domains), one can obtain a homogenized problem and prove the convergence of the solutions of the original problem towards the solution of the homogenized problem. However, these methods do not allow to estimate the rate of convergence. In the paper we use the Bakhvalov technique of asymptotic expansions [2], [3] (see also [4]) which, while requires an additional regularity of solutions, makes it possible to construct correctors and obtain error estimates. We show how it works in the second part of this material.

For simplicity in the second section we assume that the perforation is periodic and does not intersect the outer boundary of the domain. However, the approach used in the second part could be applied to the structures with more general microscopic geometry, for instance, to domains with locally periodic perforation.

A special attention will be paid to the coerciveness of operators associated with the boundary value problems studied; it turns out that it is a nontrivial issue. It will be shown that it is sufficient to verify the coerciveness of the limit operator (see for details [6] and [29]).

## 1 Appearence of a "term etrange". Dirichlet problems in domains with low concentration of holes

### 1.1 Basic notation and setting of the problem

Suppose that $\Omega$ is a domain in $\mathbb{R}^3$ with smooth boundary, $Q = \{x \ : \ |x| < 1\}$ is the unit ball in $\mathbb{R}^3$ centered in the origin. Denote

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{z \in \mathbb{Z}^3} (\varepsilon^3 Q + 2\varepsilon z),$$

i.e. $\Omega^\varepsilon$ is a domain with holes of radius $\varepsilon^3$ periodically situated with period $2\varepsilon$ (see Figure 1).

Consider the following boundary value problems:

$$\begin{cases} \Delta u_\varepsilon = f(x) \qquad \text{in} \qquad \Omega^\varepsilon, \\ u_\varepsilon \in \overset{\circ}{H}{}^1(\Omega^\varepsilon), \end{cases} \tag{1}$$

$$\begin{cases} (\Delta + \mu) u = f(x) \qquad \text{in} \qquad \Omega, \\ u \in \overset{\circ}{H}{}^1(\Omega), \ \ \mu = -\frac{\pi}{2}, \end{cases} \tag{2}$$

An interesting effect in this case is connected with the small deviation of solution $u_\varepsilon$ from the solution $u$ for sufficiently small parameter $\varepsilon$, i.e. Problem (2) is the homogenized problem for the original problem (1). Note that

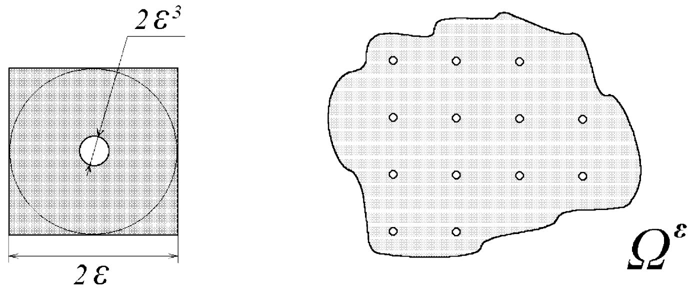


**Fig. 1.** Perforated domain

the main symbol of the operator of the homogenized problem has the same form as the main symbol of the operator of the original problem. In addition it should be noted that in the equation of the homogenized problem we have the "potential" ("term etrange").

### 1.2 The homogenization theorem

In this section we formulate and prove the basic Theorem. Denote by $w_\varepsilon$ the following $2\varepsilon$-periodic in $x$ function

$$w_\varepsilon(x) = \begin{cases} 0, & \text{if} \quad x \in \bigcup\limits_{z\in\mathbb{Z}^3} \left(\varepsilon^3 Q + 2\varepsilon z\right), \\ 1, & \text{if} \quad x \in \mathbb{R}^3 \setminus \bigcup\limits_{z\in\mathbb{Z}^3} (\varepsilon Q + 2\varepsilon z), \\ \dfrac{\left(\frac{1}{r} - \frac{1}{\varepsilon^3}\right)}{\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^3}\right)}, & \text{if} \quad x \in \bigcup\limits_{z\in\mathbb{Z}^3} \left(\varepsilon Q \setminus \varepsilon^3 Q + 2\varepsilon z\right). \end{cases} \tag{3}$$

**Theorem 1.** *For solutions $u_\varepsilon$ and $u$ to problems* (1) *and* (2) *respectively the following estimates*

$$\|u_\varepsilon - w_\varepsilon u\|_{H^1(\Omega^\varepsilon)} \le C\varepsilon \|f\|_{C^\alpha(\overline{\Omega})}, \qquad \|u_\varepsilon - u\|_{L_2(\Omega^\varepsilon)} \le C\varepsilon \|f\|_{C^\alpha(\overline{\Omega})}$$

*are valid, where $w_\varepsilon(x)$ is defined in* (3) *and $C, \alpha = \mathrm{const} > 0$, $C$ does not depend on $\varepsilon$ and $f(x)$.*

*Proof.* Using the explicit formula (3) for $w_\varepsilon$ one can check that if $\varepsilon \to 0$, then

$$\|w_\varepsilon - 1\|_{L_2(\Omega^\varepsilon)} \to 0,$$

$$\|\nabla w_\varepsilon\|_{L_2(\Omega^\varepsilon)} \to 0$$

and the following estimates

$$\|w_\varepsilon - 1\|_{L_2(\Omega^\varepsilon)} \le C_1 \varepsilon^2, \tag{4}$$

$$\|\nabla w_\varepsilon\|_{L_2(\Omega^\varepsilon)} \le C_2\varepsilon \tag{5}$$

hold true, where $C_1$ and $C_2$ do not depend on $\varepsilon$. Moreover, using the standard definition of the norm in the space $H^{-1}(\Omega^\varepsilon)$ and formula (3), one can prove that for any smooth function $u$ the estimate

$$\|(\Delta w_\varepsilon - \mu)u\|_{H^{-1}(\Omega^\varepsilon)} \le C_3\varepsilon\|u\|_{\overset{\circ}{H}{}^1(\Omega)} \tag{6}$$

is valid, where the constant $C_3 > 0$ is independent of $\varepsilon$ and $u \in \overset{\circ}{H}{}^1(\Omega)$.

Using the integral identities for Problems (1) and (2), we estimate the following integral:

$$\begin{aligned}
&\Big|\int\limits_{\Omega^\varepsilon} \nabla(u_\varepsilon - w_\varepsilon u)\nabla\psi\ dx\Big| \\
&= \Big|\int\limits_{\Omega^\varepsilon} \nabla u_\varepsilon\nabla\psi\ dx - \int\limits_{\Omega^\varepsilon} u\nabla w_\varepsilon\nabla\psi\, dx - \int\limits_{\Omega^\varepsilon} w_\varepsilon\nabla u\nabla\psi\, dx\Big| \\
&= \Big|-\int\limits_{\Omega^\varepsilon} f\psi\ dx - \int\limits_{\Omega^\varepsilon} u\nabla w_\varepsilon\nabla\psi\ dx - \int\limits_{\Omega} \nabla u\nabla\psi\, dx - \int\limits_{\Omega}(w_\varepsilon - 1)\nabla u\nabla\psi\, dx\ \Big| \\
&\le \Big|\int\limits_{\Omega\setminus\Omega^\varepsilon} f\psi\ dx\Big| + \Big|\int\limits_{\Omega^\varepsilon} u\nabla w_\varepsilon\nabla\psi\ dx + \int\limits_{\Omega}\mu u\psi\, dx\Big| + \Big|\int\limits_{\Omega}(w_\varepsilon - 1)\nabla u\nabla\psi\ dx\Big| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

It is easy to see that

$$I_1 \le C_4\varepsilon\|f\|_{C^\alpha(\overline{\Omega})}\|\psi\|_{H^1(\Omega^\varepsilon)}.$$

Integrating by parts, keeping in mind inequalities (5), (6) and the following Schauder estimate (see for instance, [18]) for the solution to Problem (2):

$$\|u\|_{C^{\alpha+2}(\overline{\Omega})} \le C_5\|f\|_{C^\alpha(\overline{\Omega})},$$

where $\alpha > 0, C_5 > 0$ are constants and $C_5 > 0$ does not depend on $f(x)$, one can estimate $I_2$ as follows:

$$I_2 \le C_6\varepsilon\|f\|_{C^\alpha(\overline{\Omega})}\|\psi\|_{H^1(\Omega^\varepsilon)}.$$

By (4) we obtain

$$I_3 \le C_7\varepsilon\|f\|_{C^\alpha(\overline{\Omega})}\|\psi\|_{H^1(\Omega^\varepsilon)}$$

and finally

$$\Big|\int\limits_{\Omega^\varepsilon} \nabla(u_\varepsilon - w_\varepsilon u)\nabla\psi\ dx\Big| \le C_8\varepsilon\|f\|_{C^\alpha(\overline{\Omega})}\|\psi\|_{H^1(\Omega^\varepsilon)} \tag{7}$$

Substituting $\psi = u_\varepsilon - w_\varepsilon u$ in (7) and using the Friedrichs inequality

$$\int\limits_{\Omega^\varepsilon} \psi^2\,dx \le C_9 \int\limits_{\Omega^\varepsilon} |\nabla\psi|^2\,dx,$$

we get the first inequality of the Theorem.

The second inequality of Theorem 1 is a consequence of the first one if we keep in mind the definition of $w_\varepsilon$. Now we complete the proof of the Theorem.

## 2 Homogenization problems in perforated domain with oscillating Fourier boundary conditions

### 2.1 Basic notation

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d \ge 2$, with a smooth boundary. Consider the sets

$$J^\varepsilon = \Big\{ j \in \mathbb{Z}^d \,:\, \operatorname{dist}(\varepsilon j, \partial\Omega) \ge \varepsilon\sqrt{d} \Big\},\ \Xi \equiv \{\xi \,|\, -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1,\dots,d\Big\}.$$

Here $\Xi$ is the periodicity cell. Given $F(\xi)$, a $\Xi$-periodic smooth function, such that

$$F(\xi)\Big|_{\xi\in\partial\Xi} \ge \text{const} > 0, \qquad F(0) = -1, \qquad \nabla_\xi F \ne 0 \quad \text{as} \quad \xi \in \Xi\backslash\{0\},$$

we define

$$Q_j^\varepsilon = \{x \in \varepsilon\,(\Xi + j) \,|\, F(\frac{x}{\varepsilon}) \le 0\}$$

and consider a periodically perforated domain

$$\Omega^\varepsilon = \Omega \backslash \bigcup_{j\in J^\varepsilon} Q_j^\varepsilon.$$

Here and everywhere below, $\Xi$-periodicity means 1-periodicity with respect to $\xi_1, \dots, \xi_d$.

Thus, the boundary $\partial\Omega^\varepsilon$ consists of $\partial\Omega$ and the boundary of inclusions $S_\varepsilon \subset \Omega$, $S_\varepsilon = (\partial\Omega^\varepsilon) \cap \Omega$. Let us denote by $Q = \{\xi \,|\, -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1,\dots,d,\ F(\xi) \le 0\}$ the inclusion (hole), by $S = \{\xi \,|\, F(\xi) = 0\}$ the boundary of $Q$, and by $\nu$ the internal unit normal to $S$ in the extended coordinates $\xi$. Note that in this section we have more general micro-inhomogeneous structure. Here $Q$ is not a ball.

In what follows the summation over repeated indices is assumed.

Consider the following boundary value problem:

$$\begin{cases} -\mathcal{L}_\varepsilon u_\varepsilon := \frac{\partial}{\partial x_k}\left(a_{kj}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}\right) = f(x) \quad \text{in} \quad \Omega^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \gamma} + p(\frac{x}{\varepsilon})u_\varepsilon + \varepsilon q(\frac{x}{\varepsilon})u_\varepsilon = g(\frac{x}{\varepsilon}) \quad \text{on} \quad S_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{8}$$

where $\frac{\partial u_\varepsilon}{\partial \gamma} := a_{kj} \frac{\partial u_\varepsilon}{\partial x_j} \nu_k^\varepsilon$ and $\nu^\varepsilon = (\nu_1^\varepsilon, \dots \nu_d^\varepsilon)$ is the unit inner normal to the boundary of the inclusions. We assume that all the functions $a_{kj}(\xi)$, $p(\xi)$, $q(\xi)$ and $g(\xi)$ are $\Xi$-periodic, the matrix $(a_{kj})$ is symmetric, and positively defined, that is

$$\kappa_1 \eta^2 \le a_{kj} \eta_k \eta_j \le \kappa_2 \eta^2 \quad \text{for any vector} \quad \eta,$$

where $0 < \kappa_1 < \kappa_2 < \infty$.

We suppose that

$$< p(\xi) >_S = < g(\xi) >_S = 0, \tag{9}$$

where $< \cdot >_S := \int\limits_S \cdot \, d\sigma$.

For a domain $G$ let $H^1(G)$ be the Sobolev space formed by all functions in $L^2(G)$ whose gradients belong to $L^2(G)$. Let us denote by $\overset{\circ}{H}{}^1(\Omega^\varepsilon, \partial\Omega)$ the closure in $H^1(\Omega^\varepsilon)$ of the set of smooth functions vanishing on $\partial\Omega$. Let $H_0^{-1}(\Omega^\varepsilon, \partial\Omega)$ be the dual space to $\overset{\circ}{H}{}^1(\Omega^\varepsilon, \partial\Omega)$ with respect to the $L^2(\Omega^\varepsilon)$ inner product. We denote by $A^\varepsilon : \overset{\circ}{H}{}^1(\Omega^\varepsilon, \partial\Omega) \to H_0^{-1}(\Omega^\varepsilon, \partial\Omega)$ the operator associated with (8).

### 2.2 Formal asymptotic analysis of the problem

We search for the solution to (8) in the standard form [4]

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1(x, \xi) + \varepsilon^2 u_2(x, \xi) + \dots, \quad \xi = \frac{x}{\varepsilon}, \tag{10}$$

where the functions $u_i(x, \xi)$ are assumed to be $\Xi$-periodic with respect to $\xi$. We will use the following notation (see [4])

$$-\mathcal{L}_{\alpha\beta} \varphi(x, \xi) := \frac{\partial}{\partial \alpha_k} \left( a_{kj}(\xi) \frac{\partial \varphi(x, \xi)}{\partial \beta_j} \right), \quad \frac{\partial \varphi(x, \xi)}{\partial \gamma_\alpha} := a_{kj}(\xi) \frac{\partial \varphi(x, \xi)}{\partial \alpha_j} \nu_k.$$

Substituting (10) in (8) and collecting the terms of order $\varepsilon^{-1}$ in the equation and of order $\varepsilon^0$ in the boundary condition on $S$ (the highest order terms) we arrive at the following auxiliary problem:

$$\begin{cases} \mathcal{L}_{\xi\xi} u_1 + \mathcal{L}_{\xi x} u_0 = 0 \quad \text{in} \quad \Xi \backslash Q, \\ \frac{\partial u_1}{\partial \gamma_\xi} + \frac{\partial u_0}{\partial \gamma_x} + p(\xi) u_0 = g(\xi) \quad \text{on} \quad S. \end{cases} \tag{11}$$

The associated integral identity takes the form

$$\int\limits_{\Xi \backslash Q} a_{kj} \frac{\partial u_1}{\partial \xi_j} \frac{\partial v}{\partial \xi_k} d\xi + \int\limits_{\Xi \backslash Q} a_{kj} \frac{\partial u_0}{\partial x_j} \frac{\partial v}{\partial \xi_k} d\xi + \int\limits_S p(\xi) u_0 v \, d\sigma = \int\limits_S g(\xi) v \, d\sigma, \tag{12}$$

where $v \in H^1_{\text{per}}(\Xi \backslash Q)$. Integral identity (12) suggests to look for the function $u_1(x, \xi)$ in the form

$$u_1(x,\xi) = L(\xi) + M(\xi)u_0(x) + N_i(\xi)\frac{\partial u_0(x)}{\partial x_i}. \tag{13}$$

Substituting the latter expression in (12) and collecting corresponding terms lead us to the following problems for functions $N_i(\xi)$, $M(\xi)$, and $L(\xi)$:

$$\int\limits_{\Xi\backslash Q} a_{kj}\frac{\partial N_i}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}d\xi + \int\limits_{\Xi\backslash Q} a_{ki}\frac{\partial v}{\partial \xi_k}d\xi = 0 \tag{14}$$

or, equivalently,

$$\begin{cases} \mathcal{L}_{\xi\xi}\left(N_i(\xi)+\xi_i\right) = 0 \quad \text{in} \quad \Xi\backslash Q, \\ \frac{\partial N_i(\xi)}{\partial \gamma_\xi} = a_{ki}(\xi)\nu_k \quad \text{on} \quad S, \end{cases}$$

where $i = 1, \ldots, d$;

$$\int\limits_{\Xi\backslash Q} a_{kj}\frac{\partial M}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}d\xi + \int\limits_{S} p(\xi)v\ d\sigma = 0; \tag{15}$$

$$\int\limits_{\Xi\backslash Q} a_{kj}\frac{\partial L}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}d\xi = \int\limits_{S} g(\xi)v\ d\sigma. \tag{16}$$

It is easy to verify that (9) is the solvability condition for problems (15) and (16). Note that the functions $L(\xi)$, $M(\xi)$, and $N_i(\xi)$ are defined up to an additive constant that can be fixed by the following normalization condition:

$$<L>_{\Xi\backslash Q} = <M>_{\Xi\backslash Q} = <N_i>_{\Xi\backslash Q} = 0 \qquad \forall\ i = 1, \ldots, d,$$

which is assumed to be fulfilled later on.

Similarly, collecting the terms of order $\varepsilon^0$ in the equation and of order $\varepsilon^1$ in the boundary condition on $S$ leads to the following problem for $u_2(x,\xi)$:

$$\begin{cases} \mathcal{L}_{\xi\xi}u_2 + \mathcal{L}_{x\xi}u_1 + \mathcal{L}_{\xi x}u_1 + \mathcal{L}_{xx}u_0 = -f \quad \text{in} \quad \Xi\backslash Q, \\ \frac{\partial u_2}{\partial \gamma_\xi} + \frac{\partial u_1}{\partial \gamma_x} + p(\xi)u_1 + q(\xi)u_0 = 0 \quad \text{on} \quad S. \end{cases} \tag{17}$$

We need the following result.

**Lemma 1.** *The functions $M(\xi)$ and $N_k(\xi)$ satisfy*

$$\frac{\partial u_0(x)}{\partial x_k}\Big(\int\limits_{\Xi\backslash Q} a_{kj}\frac{\partial M}{\partial \xi_j}\ d\xi - \int\limits_{S} pN_k\ d\sigma\Big) = 0.$$

*Proof.* Substituting $N_i(\xi)$ as a test function in (15) gives

$$\int\limits_{\Xi\backslash Q} a_{kj}\frac{\partial M}{\partial \xi_j}\frac{\partial N_i}{\partial \xi_k}d\xi + \int\limits_{S} p(\xi)N_i\ d\sigma = 0.$$

Substituting $M(\xi)$ as a test function in (14) gives

$$\int\limits_{\Xi\setminus Q} a_{kj}\frac{\partial N_i}{\partial \xi_j}\frac{\partial M}{\partial \xi_k}d\xi + \int\limits_{\Xi\setminus Q} a_{ki}\frac{\partial M}{\partial \xi_k}d\xi = 0.$$

Since the matrix $(a_{ij})$ is symmetric, we get

$$\int\limits_{\Xi\setminus Q} a_{kj}\frac{\partial M}{\partial \xi_j}\, d\xi = \int\limits_{S} pN_k\, d\sigma.$$

The lemma is proved. □

The weak formulation of problem (17) reads

$$\int\limits_{\Xi\setminus Q} a_{kj}\frac{\partial u_2}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}d\xi + \int\limits_{\Xi\setminus Q} a_{kj}\frac{\partial u_1}{\partial x_j}\frac{\partial v}{\partial \xi_k}d\xi + \int\limits_{S} p(\xi)u_1 v\, d\sigma + u_0(x)\int\limits_{S} q(\xi)v\, d\sigma$$

$$-\int\limits_{\Xi\setminus Q} a_{kj}\frac{\partial M}{\partial \xi_j}v d\xi\frac{\partial u_0}{\partial x_k} - \int\limits_{\Xi\setminus Q}\Big(a_{ij}\frac{\partial N_k}{\partial \xi_j} + a_{ik}\Big)v d\xi\frac{\partial^2 u_0}{\partial x_i\partial x_k} + |\Xi\setminus Q| f = 0$$

for any $v \in H^1_{\rm per}(\Omega^\varepsilon)$. Now, in order to obtain the desired homogenized equation, we write down the solvability condition for problem (17). By applying the above lemma, this equation can be rewritten as follows:

$$\widehat{a}_{kj}\frac{\partial^2 u_0(x)}{\partial x_k\partial x_j} - u_0(x)\Big(\int\limits_S q(\xi)\, d\sigma + \int\limits_S p(\xi)M(\xi)\, d\sigma\Big)$$

$$= |\Xi\setminus Q| f(x) + \int\limits_S g(\xi)M(\xi)\, d\sigma, \tag{18}$$

where

$$\widehat{a}_{ik} := \int\limits_{\Xi\setminus Q}\Big(a_{ij}(\xi)\frac{\partial N_k(\xi)}{\partial \xi_j} + a_{ik}(\xi)\Big)d\xi.$$

Finally, the homogenized problem reads

$$\begin{cases} \widehat{a}_{kj}\dfrac{\partial^2 u_0(x)}{\partial x_k\partial x_j} - <q>_S u_0(x) + m\, u_0(x) = |\Xi\setminus Q| f(x) - l \quad \text{in} \quad \Omega, \\ u_0(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{19}$$

where

$$m := -<pM>_S, \qquad l := <pL>_S = -<gM>_S.$$

Let $\widehat{A} : \overset{\circ}{H}{}^1(\Omega, \partial\Omega) \to H^{-1}(\Omega, \partial\Omega)$ be the operator associated with the limit problem (19).

*Remark 1.* The coerciveness of the limit problem (19) is a delicate question since the constant $m$, as we will see later, is always positive. In particular, the coerciveness of (19) is provided by the inequality

$$m - <q>_S < \lambda_0,$$

where $\lambda_0$ is the first eigenvalue of the differential operator $-\widehat{a}_{ij}\dfrac{\partial^2}{\partial x_i \partial x_j}$ in the Sobolev space $\overset{\circ}{H}{}^1(\Omega, \partial\Omega)$.

### 2.3 Main estimates and results

In this section we obtain upper and lower bounds for the coefficient $m$ in (19) and then formulate the main statement of the paper.

We begin by considering an auxiliary spectral problem of the Steklov type

$$\begin{cases} \dfrac{\partial}{\partial \xi_k}\left(a_{kj}(\xi)\dfrac{\partial \theta}{\partial \xi_j}\right) = 0 \quad \text{in} \quad \Xi \backslash Q, \\ \dfrac{\partial \theta}{\partial \gamma} = \Upsilon \theta \quad \text{on} \quad S, \\ \theta(\xi) \quad \text{is } \Xi\text{-periodic in} \quad \xi, \qquad <\theta>_S = 0, \end{cases} \tag{20}$$

where $\Upsilon$ is a spectral parameter. The first eigenvalue $\Upsilon_1$ of problem (20) can be found from the variational principle

$$\Upsilon_1 = \inf\left\{ \frac{a(\psi,\psi)}{<\psi^2>_S} : \psi \in H^1_{\text{per}}(\Xi)\backslash\{0\},\ <\psi>_S = 0 \right\},$$

where $a(u,v) := \displaystyle\int_{\Xi\backslash Q} a_{kj}\frac{\partial u}{\partial \xi_j}\frac{\partial v}{\partial \xi_k}\, d\xi.$

The following lemma provides us with estimates of the coefficient $m$ in the homogenized problem (19).

**Lemma 2.** *The constant $m$ is positive. Moreover, it satisfies the estimates*

$$<p^2>_S \ \frac{<p^2>_S}{a(p,p)} \ \le \ m \ \le \ \frac{<p^2>_S}{\Upsilon_1}. \tag{21}$$

*Remark 2.* Note that the equalities in (21) are only attained if $p(\xi)$ happens to be the first eigenfunction in (20), i.e. the eigenfunction that corresponds to the eigenvalue $\Upsilon_1$.

*Proof.* Substituting $M(\xi)$ as a test function in (15) we have

$$\int_{\Xi\backslash Q} a_{kj}\frac{\partial M}{\partial \xi_j}\frac{\partial M}{\partial \xi_k} d\xi + \int_S p(\xi) M \; d\sigma = 0.$$

Thus,

$$m = - <pM>_S = \left\langle a_{kj} \frac{\partial M}{\partial \xi_j} \frac{\partial M}{\partial \xi_k} \right\rangle_{\Xi \setminus Q} > 0,$$

if $M \not\equiv 0$. Note that $M \equiv 0$ if $p \equiv 0$.

Consider a variational problem

$$\inf_{\psi \in H^1_{\mathrm{per}}(\Xi)} H(\psi) \equiv \inf_{\psi \in H^1_{\mathrm{per}}(\Xi)} \left\{ a(\psi, \psi) + 2 <p\psi>_S \right\}. \tag{22}$$

It follows from (15) that $M(\xi)$ provides the infimum in (22). Therefore,

$$- \inf_{\psi \in H^1_{\mathrm{per}}(\Xi)} \left\{ a(\psi, \psi) + 2 <p\psi>_S \right\}$$

$$= -a(M, M) + 2 <pM>_S = - <pM>_S = m.$$

Substituting $\psi = -tp$ in the functional $H(\psi)$, we get

$$H(-tp) = t^2 a(p,p) - 2t <p^2>_S.$$

To locate the minimum of $H(-tp)$ in $t$, we solve the equation $0 = H'_t(-t_0 p) = 2t_0 a(p,p) - 2 <p^2>_S$. This gives

$$t_0 = \frac{<p^2>_S}{a(p,p)},$$

and, hence,

$$H(-t_0 p) = \frac{\left(<p^2>_S\right)^2}{a(p,p)} - 2\frac{\left(<p^2>_S\right)^2}{a(p,p)} = - \frac{\left(<p^2>_S\right)^2}{a(p,p)}.$$

This completes the proof of the first inequality in (21).

Similarly, locating the minimum of $H(-t\varphi)$ in $t$ for an arbitrary function $\varphi$, we get

$$H(-t_0\varphi) = - \frac{\left(<p\varphi>_S\right)^2}{a(\varphi,\varphi)}.$$

Since

$$m = \sup_{\varphi \in H^1_{\mathrm{per}}(\Xi)} \frac{\left(<p\varphi>_S\right)^2}{a(\varphi,\varphi)}, \tag{23}$$

and the function $\varphi = M$ brings the supremum in (23), then for an arbitrary $\varphi$ we have

$$m \geq \frac{\left(<p\varphi>_S\right)^2}{a(\varphi,\varphi)}.$$

It follows from (23) that

$$\frac{1}{m} = \inf_{\varphi \in H^1_{\rm per}(\Xi)\setminus\{0\}} \frac{a(\varphi,\varphi)}{\left(<p\varphi>_S\right)^2} \geq \inf_{\substack{\varphi \in H^1_{\rm per}(\Xi)\setminus\{0\},\\ <\varphi>_S=0}} \frac{a(\varphi,\varphi)}{<p^2>_S\ <\varphi^2>_S} = \frac{\Upsilon_1}{<p^2>_S}.$$

Finally,

$$m \leq \frac{<p^2>_S}{\Upsilon_1},$$

where $\Upsilon_1$ is the first eigenvalue of the spectral problem (20). This proves the second inequality in (21). The lemma is proved. □

*Remark 3.* To give a simple explanation of the positiveness of the coefficient $m$ arising in the homogenized problem (19), we consider the operator associated with the following modification of (8): $q \equiv 0$ and instead of the Dirichlet boundary condition on the exterior boundary $\partial\Omega$ the Neumann boundary condition is stated. In this case $-m$ is the first eigenvalue of the operator associated with the respective homogenized problem. In view of the convergence of the spectrum, the first eigenvalue of the operator introduced above is close to $-m$ for sufficiently small $\varepsilon$. The substitution of a constant function in the variational principle for the first eigenvalue of this operator gives zero. Thus, for any $\varepsilon$ the said eigenvalue is negative and so is $-m$.

The following theorem gives an $O(\sqrt{\varepsilon})$ estimate of the discrepancy between a solution to (8) and two leading terms of expansion (10), in the Sobolev space $H^1$.

**Theorem 2.** *Let $f(x) \in C^1(\Omega)$, and suppose that $p(\xi)$, $q(\xi)$, and $g(\xi)$ are $\Xi$-periodic $C^1$ functions. Furthermore, assume that*

$$m < \lambda_0 + <q>_S, \tag{24}$$

*where $\lambda_0$ is defined in* Remark 1. *Then for all sufficiently small $\varepsilon > 0$ problem* (8) *has a unique solution $u_\varepsilon(x)$ and the following estimate holds*

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq K_1\sqrt{\varepsilon} \tag{25}$$

*with a constant $K_1 > 0$ independent of $\varepsilon$. Here $u_0$ is the solution to* (19) *and $u_1$ is defined by* (13) *with functions $L(\xi)$, $M(\xi)$ and $N_i(\xi)$ constructed in* (14), (15), *and* (16) *respectively.*

### 2.4 Auxiliary results

This section is devoted to various technical results used in further analysis. The first two statements can be found in [5] and [9] and are formulated here for the sake of completeness. Their proof is omitted. Other lemmas can be found in [6].

**Lemma 3.** *If*

$$\frac{1}{|\Xi \cap \omega|} \int\limits_{\Xi\cap\omega} <q>_S \, d\xi - \int\limits_{S} q(x,\xi)\ d\sigma \equiv 0, \tag{26}$$

*then the following estimate holds*

$$\Big| \frac{1}{|\Xi \cap \omega|} \int\limits_{\Omega^\varepsilon} <q>_S u(x)v(x)\,dx - \varepsilon \int\limits_{S_\varepsilon} q(x, \frac{x}{\varepsilon})\, u(x)v(x)\ ds \Big| \tag{27}$$

$$\le C_2 \varepsilon \|u\|_{H^1(\Omega^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}.$$

*for any* $u(x), v(x) \in H^1(\Omega^\varepsilon)$ *with constant* $C_2$ *independent of* $\varepsilon$.

The next lemma allows us to neglect the right hand-side of (8) in the layer $\Pi_\varepsilon := \{x \in \Omega \ :\ \operatorname{dist}(x, \partial\Omega) \le \varepsilon\sqrt{d}\}$ without worsening the estimate (25).

**Lemma 4.** *Let* $y_\varepsilon$ *be the solution to*

$$\begin{cases} \mathcal{L}_\varepsilon y_\varepsilon = -h^\varepsilon(x) \quad in \quad \Omega^\varepsilon, \\ \frac{\partial y_\varepsilon}{\partial \gamma} + p(\frac{x}{\varepsilon}) y_\varepsilon + \varepsilon q(\frac{x}{\varepsilon}) y_\varepsilon = g(\frac{x}{\varepsilon}) \quad on \quad S_\varepsilon, \\ y_\varepsilon = 0 \quad on \quad \partial\Omega, \end{cases} \tag{28}$$

*where* $h^\varepsilon(x) = f(x)$ *in* $\Pi_\varepsilon$ *and* 0 *in the remaining part of* $\Omega^\varepsilon$. *Then*

$$\|y_\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C_3 \varepsilon. \tag{29}$$

Let $\lambda_0$ be the first eigenvalue of the homogenized operator (it is defined in Remark 1). The statement below characterizes the coerciveness property of the limit problem.

**Lemma 5.** *If* $m < \lambda_0 + <q>_S$ *then problem* (19) *is coercive.*

*Proof.* The variational principle for the first eigenvalue of the operator $\widehat{A}$ leads to the following relation

$$\inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega), \\ \|v\|_{L_2(\Omega)}=1}} \left(-\widehat{A}v, v\right)_{L_2(\Omega)} = \inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega), \\ \|v\|_{L_2(\Omega)}=1}} \int\limits_{\Omega} \widehat{a}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + \big(<q>_S - m\big)\ v^2\,dx$$

$$= \inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega), \\ \|v\|_{L_2(\Omega)}=1}} \int\limits_{\Omega} \widehat{a}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\,dx + \big(<q>_S - m\big) = \lambda_0 + <q>_S - m.$$

Thus

$$\left(-\widehat{A}v, v\right)_{L_2(\Omega)} \ge\ C_4\ \|v\|^2_{L_2(\Omega)}, \qquad C_4 > 0,$$

that completes the proof. □

The next lemma can be considered as a modified version of Lemma 3. The functions $u(\xi)$ and $v(\xi)$ in its formulation are not assumed to be periodic.

**Lemma 6.** *If* $<p>_S = 0$ *then*

$$\Big|\int_S p(\xi)\, u(\xi)v(\xi)\; d\sigma\Big| \le C_5 \left(\|\nabla u\|_{L_2(\Xi)}\|v\|_{L_2(\Xi)} + \|u\|_{L_2(\Xi)}\|\nabla v\|_{L_2(\Xi)}\right) \quad (30)$$

*for any* $u(\xi), v(\xi) \in H^1(\Xi)$. *Here* $C_5$ *is a constant independent of* $\varepsilon$.

*Proof.* Since $<p>_S = 0$, the problem

$$\begin{cases} \Delta_\xi \Psi(\xi) = 0 \quad \text{in} \quad \Xi\backslash Q, \\ \dfrac{\partial \Psi}{\partial n} = p(\xi) \quad \text{on} \quad S, \qquad \dfrac{\partial \Psi}{\partial n} = 0 \quad \text{on} \quad \partial\Xi \end{cases} \quad (31)$$

has a unique solution up to an additive constant.

Let us multiply the first equation in (31) by $u(\xi)v(\xi)$ where $u(\xi), v(\xi) \in H^1(\Xi)$, and integrate it over $\Xi\backslash Q$. Integration by parts on the left hand-side gives

$$\begin{aligned} \Big|\int_S p(\xi)\, u(\xi)v(\xi)\; d\sigma\Big| &= \Big|\int_{\Xi\backslash Q} \Delta_\xi \Psi(\xi) u(\xi)v(\xi)\; d\xi - \int_S p(\xi)\, u(\xi)v(\xi)\; d\sigma\Big| \\ &\le \Big|\int_{\Xi\backslash Q} \left((\nabla_\xi \Psi(\xi)), \nabla_\xi \left(u(\xi)v(\xi)\right)\right) d\xi\Big| \qquad (32) \\ &\le C_5 \left(\|\nabla u\|_{L_2(\Xi)}\|v\|_{L_2(\Xi)} + \|u\|_{L_2(\Xi)}\|\nabla v\|_{L_2(\Xi)}\right). \end{aligned}$$

The lemma is proved. □

The next lemma shows that the bilinear form associated with problem (8) is coercive uniformly in $\varepsilon$. In particular, this proves that (8) is well-posed for sufficiently small $\varepsilon$.

**Lemma 7.** *The coerciveness of the homogenized problem* (19) *implies the coerciveness of the original problem* (8) *for all sufficiently small* $\varepsilon$.

*Proof.* First, we show that

$$\int_{S_\varepsilon} p(\frac{x}{\varepsilon}) u^2(x)\; ds \le \alpha \int_{\Omega^\varepsilon} |\nabla u|^2\; dx + \frac{1}{\alpha}\int_{\Omega^\varepsilon} u^2\, dx \quad (33)$$

for any $\alpha > 0$. Indeed, by Lemma 6 we get

$$\Big|\int_S p(\xi)\, u^2(\xi)\, d\sigma\Big| \le 2C_5 \|\nabla u\|_{L_2(\Xi)}\|u\|_{L_2(\Xi)} \le C_6 (\frac{\alpha}{\varepsilon}\int_\Xi |\nabla u|^2\, d\xi + \frac{\varepsilon}{\alpha}\int_\Xi u^2\, d\xi).$$

Now rewriting the latter inequality in the coordinates $x = \varepsilon\xi$ and summing up over all the periodicity cells inside $\Omega$ we obtain (33).

The next step is to verify that there exists a sufficiently large $\Lambda$ independent of $\varepsilon$ such that the operator associated with the boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon u_\varepsilon + \Lambda u_\varepsilon = -f(x) \quad \text{in} \quad \Omega^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \gamma} + p(\frac{x}{\varepsilon})u_\varepsilon + \varepsilon q(\frac{x}{\varepsilon})u_\varepsilon = g(\frac{x}{\varepsilon}) \quad \text{on} \quad S_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{34}$$

is coercive for any $\varepsilon > 0$.

It follows from Lemma 3 that

$$\int\limits_{\Omega^\varepsilon} a_{ik}(\frac{x}{\varepsilon})\frac{\partial v}{\partial x_k}\frac{\partial v}{\partial x_i}dx + \int\limits_{S_\varepsilon} p(\frac{x}{\varepsilon})v^2 ds + \varepsilon \int\limits_{S_\varepsilon} q(\frac{x}{\varepsilon})v^2 ds + \int\limits_{\Omega^\varepsilon} \Lambda v^2\, dx$$

$$\geq \kappa_1 \|\nabla v\|^2_{L_2(\Omega^\varepsilon)} - (C_7\alpha + O(\varepsilon))\|\nabla u\|^2_{L_2(\Omega^\varepsilon)} + \left(<q>_S - \frac{C_7}{\alpha} + \Lambda\right)\|v\|^2_{L_2(\Omega^\varepsilon)}. \tag{35}$$

By taking a sufficiently small $\alpha$ and, then, a sufficiently large $\Lambda$, we establish the positive definiteness of the bilinear form on the right hand-side of (35) and, therefore, the coerciveness.

Consider the following spectral problems

$$(A^\varepsilon + \Lambda \cdot 1)^{-1} u^k_\varepsilon = \mu^\varepsilon_k u^k_\varepsilon, \tag{36}$$

$$\left(\widehat{A} + \Lambda \cdot 1\right)^{-1} u^k = \mu_k u^k, \tag{37}$$

where $A^\varepsilon$ is the operator associated with the initial problem (8) and $\widehat{A}$ is the operator associated with the homogenized problem (19).

Taking into account the coerciveness shown above, it is easy to verify that the operators $(A^\varepsilon + \Lambda \cdot 1)^{-1}$ and $\left(\widehat{A} + \Lambda \cdot 1\right)^{-1}$ satisfy conditions $C1$–$C4$ of Theorem 1.9 from [25, Chapter III]. In particular, Theorem 1.9 implies that $\mu^\varepsilon_0 \to \mu_0$ as $\varepsilon \to 0$. Let us denote by $\lambda^\varepsilon_0$ and $\lambda_0$ the first eigenvalues of the operators $A^\varepsilon$ and $\widehat{A}$ respectively. Then

$$\lambda^\varepsilon_0 \equiv -\Lambda + \frac{1}{\mu^\varepsilon_0} \to \lambda_0 \equiv -\Lambda + \frac{1}{\mu_0} \qquad \text{as} \quad \varepsilon \to 0.$$

Thus, the positive definiteness of $\widehat{A}$ implies the positive definiteness of $A^\varepsilon$ for all sufficiently small $\varepsilon$. The lemma is proved. □

*Remark 4.* The analogous approach can be found in [29] (Lemma 3) and, for instance, the coercivity of Problem (34) follows from the results of this paper.

### 2.5 Justification of asymptotics

*Proof of Theorem 2.* To estimate the $H^1$-norm of the remainder

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)}$$

let us substitute the expression $z_\varepsilon(x, \frac{x}{\varepsilon}) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) - u_\varepsilon(x)$ in (8). This gives

$$\mathcal{L}_\varepsilon \left( z_\varepsilon(x, \frac{x}{\varepsilon}) \right) = \frac{1}{\varepsilon} \mathcal{L}_{\xi x} u_0(x) \Big|_{\xi = \frac{x}{\varepsilon}} + \mathcal{L}_\varepsilon u_0(x) + \varepsilon \mathcal{L}_{xx} u_1(x, \xi) \Big|_{\xi = \frac{x}{\varepsilon}}$$

$$+ \mathcal{L}_{\xi x} u_1(x, \xi) \Big|_{\xi = \frac{x}{\varepsilon}} + \mathcal{L}_{x \xi} u_1(x, \xi) \Big|_{\xi = \frac{x}{\varepsilon}} + \frac{1}{\varepsilon} \mathcal{L}_{\xi \xi} u_1(x, \xi) \Big|_{\xi = \frac{x}{\varepsilon}} - \mathcal{L}_\varepsilon u_\varepsilon(x). \quad (38)$$

Taking into account the relations

$$\mathcal{L}_{\xi \xi} u_1(x, \xi) = -\mathcal{L}_{\xi x} u_0(x), \qquad \mathcal{L}_\varepsilon u_\varepsilon(x) = -f(x),$$

$$-\mathcal{L}_{\xi x} u_1(x, \xi) = \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_j} M(\xi) \right) + \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \, \partial x_k} N_k(\xi) \right),$$

$$-\mathcal{L}_{x \xi} u_1(x, \xi) = a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_i} \frac{\partial M(\xi)}{\partial \xi_j} + a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_i \, \partial x_k} \frac{\partial N_k(\xi)}{\partial \xi_j}, \quad (39)$$

and

$$\widehat{a}_{kj} \frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} - {<q>}_S \, u_0(x) + m \, u_0(x) = |\Xi \backslash Q| f(x) - l \quad \text{in} \quad \Omega, \quad (40)$$

one can rewrite (38) in the domain $\Omega^\varepsilon$ as follows

$$-\mathcal{L}_\varepsilon \left( z_\varepsilon(x, \frac{x}{\varepsilon}) \right) = -\varepsilon \mathcal{L}_{xx} u_1(x, \xi) \Big|_{\xi = \frac{x}{\varepsilon}} + \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_j} M(\xi) \right) \Big|_{\xi = \frac{x}{\varepsilon}}$$

$$+ \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \, \partial x_k} N_k(\xi) \right) \Big|_{\xi = \frac{x}{\varepsilon}}$$

$$+ a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_i} \frac{\partial M(\xi)}{\partial \xi_j} \Big|_{\xi = \frac{x}{\varepsilon}} + a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_i \, \partial x_k} \frac{\partial N_k(\xi)}{\partial \xi_j} \Big|_{\xi = \frac{x}{\varepsilon}} - \mathcal{L}_\varepsilon u_0(x) \quad (41)$$

$$- \frac{1}{|\Xi \cap \omega|} \hat{a}_{kj} \frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} + \frac{({<q>}_S - m)}{|\Xi \cap \omega|} \, u_0(x) - \frac{l}{|\Xi \cap \omega|}.$$

Similarly, we have on $S_\varepsilon$

$$\frac{\partial z_\varepsilon(x, \frac{x}{\varepsilon})}{\partial \gamma} = -\frac{\partial u_\varepsilon(x)}{\partial \gamma_x} + \frac{\partial u_0(x)}{\partial \gamma_x} + \varepsilon \frac{\partial u_1(x, \xi)}{\partial \gamma_x} \Big|_{\xi = \frac{x}{\varepsilon}} + \frac{\partial u_1(x, \xi)}{\partial \gamma_\xi} \Big|_{\xi = \frac{x}{\varepsilon}}$$

$$= p(\frac{x}{\varepsilon})u_\varepsilon(x) + \varepsilon q(\frac{x}{\varepsilon})u_\varepsilon(x) - g(\frac{x}{\varepsilon}) + \frac{\partial u_0(x)}{\partial \gamma_x} + \varepsilon \frac{\partial u_1(x,\xi)}{\partial \gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}}$$

$$+ \frac{\partial L(\xi)}{\partial \gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}} + u_0(x)\frac{\partial M(\xi)}{\partial \gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}} + \frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(\xi)}{\partial \gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}.$$

Now multiplying (41) by $v(x)$ and integrating over $\Omega^\varepsilon$ we get

$$-\int\limits_{\Omega^\varepsilon} \mathcal{L}_\varepsilon \left(z_\varepsilon(x, \frac{x}{\varepsilon})\right) v(x)\, dx = -\varepsilon \int\limits_{\Omega^\varepsilon} \mathcal{L}_{xx} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$+ \int\limits_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_j} M(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$+ \int\limits_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_j\, \partial x_k} N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$+ \int\limits_{\Omega^\varepsilon} a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_i}\frac{\partial M(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$+ \int\limits_{\Omega^\varepsilon} a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_k}\frac{\partial N_k(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx - \int\limits_{\Omega^\varepsilon} \mathcal{L}_\varepsilon u_0(x) v(x)\, dx \tag{42}$$

$$-\frac{1}{|\Xi \cap \omega|}\int\limits_{\Omega^\varepsilon} \widehat{a}_{kj}\frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} v(x)\; dx + \frac{1}{|\Xi \cap \omega|}\int\limits_{\Omega^\varepsilon} \left(<q>_S - m\right) u_0(x)\, v(x)\; dx$$

$$-\frac{l}{|\Xi \cap \omega|}\int\limits_{\Omega^\varepsilon} v(x)\, dx.$$

On the other hand, the Green formula allows to rewrite the left hand-side of (42) as follows

$$-\int\limits_{\Omega^\varepsilon} \mathcal{L}_\varepsilon \left(z_\varepsilon(x, \frac{x}{\varepsilon})\right) v(x)\, dx = \int\limits_{S_\varepsilon} \frac{\partial z_\varepsilon}{\partial \gamma} v(x)\; ds - \int\limits_{\Omega^\varepsilon} \nabla z_\varepsilon\, \nabla v(x)\, dx$$

$$= \int\limits_{S_\varepsilon} p(\frac{x}{\varepsilon})u_\varepsilon(x)\, v(x)\; ds + \varepsilon \int\limits_{S_\varepsilon} q(\frac{x}{\varepsilon})u_\varepsilon(x)\, v(x)\; ds - \int\limits_{S_\varepsilon} g(\frac{x}{\varepsilon})v(x)\; ds$$

$$+ \int\limits_{S_\varepsilon} \frac{\partial u_0(x)}{\partial \gamma_x}\, v(x)\; ds + \varepsilon \int\limits_{S_\varepsilon} \frac{\partial u_1(x,\xi)}{\partial \gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\; ds \tag{43}$$

$$+ \int\limits_{S_\varepsilon} \left(\frac{\partial L(\xi)}{\partial \gamma_\xi} + u_0(x)\frac{\partial M(\xi)}{\partial \gamma_\xi} + \frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(\xi)}{\partial \gamma_\xi}\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\; ds$$

$$-\int\limits_{\Omega^\varepsilon} \nabla z_\varepsilon(x, \frac{x}{\varepsilon})\, \nabla v(x)\, dx.$$

From (42) and (43) we derive

$$\int\limits_{\Omega^\varepsilon} \nabla z_\varepsilon(x, \frac{x}{\varepsilon})\, \nabla v(x)\, dx = \int\limits_{S_\varepsilon} p(\frac{x}{\varepsilon}) u_\varepsilon(x)\, v(x)\ ds + \varepsilon \int\limits_{S_\varepsilon} q(\frac{x}{\varepsilon}) u_\varepsilon(x)\, v(x)\ ds$$

$$-\int\limits_{S_\varepsilon} g(\frac{x}{\varepsilon}) v(x)\ ds + \int\limits_{S_\varepsilon} \frac{\partial u_0(x)}{\partial \gamma_x}\, v(x)\ ds + \varepsilon \int\limits_{S_\varepsilon} \frac{\partial u_1(x,\xi)}{\partial \gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\ ds$$

$$+\int\limits_{S_\varepsilon} \left( \frac{\partial L(\xi)}{\partial \gamma_\xi} + u_0(x) \frac{\partial M(\xi)}{\partial \gamma_\xi} + \frac{\partial u_0(x)}{\partial x_i} \frac{\partial N_i(\xi)}{\partial \gamma_\xi} \right) \Big|_{\xi=\frac{x}{\varepsilon}} v(x)\ ds \tag{44}$$

$$-\int\limits_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_j} M(\xi) \right) \Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$-\int\limits_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j\, \partial x_k} N_k(\xi) \right) \Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$-\int\limits_{\Omega^\varepsilon} a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_i} \frac{\partial M(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx + \varepsilon \int\limits_{\Omega^\varepsilon} \mathcal{L}_{xx} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx$$

$$-\int\limits_{\Omega^\varepsilon} a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_k} \frac{\partial N_k(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx + \int\limits_{\Omega^\varepsilon} \mathcal{L}_\varepsilon u_0(x) v(x)\, dx$$

$$+\frac{1}{|\Xi \cap \omega|} \int\limits_{\Omega^\varepsilon} \widehat{a}_{kj} \frac{\partial^2 u_0(x)}{\partial x_k \partial x_j}\, v(x)\ dx - \frac{1}{|\Xi \cap \omega|} \int\limits_{\Omega^\varepsilon} \big( <q>_S - m \big)\ u_0(x)\, v(x)\ dx$$

$$+\frac{l}{|\Xi \cap \omega|} \int\limits_{\Omega^\varepsilon} v(x)\, dx.$$

In view of the obvious relation

$$\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi) \right) \Big|_{\xi=\frac{x}{\varepsilon}} = \varepsilon \frac{\partial}{\partial x_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi) \Big|_{\xi=\frac{x}{\varepsilon}} \right)$$

$$-\varepsilon \frac{\partial}{\partial x_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi) \right) \Big|_{\xi=\frac{x}{\varepsilon}},$$

we have after integration by parts,

$$\int\limits_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j\, \partial x_k} N_k(\xi) \right) \Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx \tag{45}$$

$$+\int\limits_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_j}M(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx = \varepsilon\int\limits_{S_\varepsilon} \frac{\partial u_1(x,\xi)}{\partial \gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\;ds.$$

Using (44) and the boundary condition in (11) one can obtain, after simple rearrangements, the following inequality

$$\begin{aligned}
&\Big|\int\limits_{\Omega^\varepsilon} \nabla z_\varepsilon(x,\frac{x}{\varepsilon})\,\nabla v(x)\,dx + \int\limits_{S_\varepsilon}\Big(p(\frac{x}{\varepsilon})+\varepsilon q(\frac{x}{\varepsilon})\Big)\,z_\varepsilon(x,\frac{x}{\varepsilon})\,v(x)\;ds\Big| \\
&\le \varepsilon\Big|\varepsilon\int\limits_{S_\varepsilon} q(\frac{x}{\varepsilon})u_1(x,\frac{x}{\varepsilon})\,v(x)\;ds + \int\limits_{S_\varepsilon} p(\frac{x}{\varepsilon})u_1(x,\frac{x}{\varepsilon})\,v(x)\;ds\Big| \\
&+\Big|\varepsilon\int\limits_{S_\varepsilon} q(\frac{x}{\varepsilon})u_0(x)\,v(x)\;ds - \frac{1}{|\Xi\cap\omega|}\int\limits_{\Omega^\varepsilon} < q >_S u_0(x)\,v(x)\,dx\Big| \\
&+\Big|\varepsilon\int\limits_{\Omega^\varepsilon} \mathcal{L}_{xx}u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}\,v(x)\,dx\Big| \qquad (46)\\
&+\Big|\int\limits_{S_\varepsilon}\Big(\frac{\partial u_0(x)}{\partial\gamma_x} + \frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(x,\xi)}{\partial\gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}\Big)\,v(x)\;ds\Big| \\
&+\Big|\int\limits_{S_\varepsilon} p(\frac{x}{\varepsilon})\,u_0(x)\,v(x)\;ds + \frac{1}{|\Xi\cap\omega|}\int\limits_{\Omega^\varepsilon} m u_0(x)v(x)\,dx\Big| \\
&+\Big|\frac{l}{|\Xi\cap\omega|}\int\limits_{\Omega^\varepsilon} v(x)\,d\hat{x} - \int\limits_{S_\varepsilon} g(\frac{x}{\varepsilon})\,v(x)\;ds\Big| \\
&+\Big|\int\limits_{\Omega^\varepsilon}\Big(\frac{\widehat{a}_{kj}}{|\Xi\cap\omega|}\frac{\partial^2 u_0(x)}{\partial x_k\partial x_j}\,v(x) - a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_k}\frac{\partial N_k(\xi)}{\partial\xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x) \\
&+\mathcal{L}_\varepsilon u_0(x)v(x)\Big)\,dx\Big| = I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}$$

It remains to estimate all the terms on the right hand side. We begin with $I_2$. According to Lemma 3,

$$\begin{aligned}
I_2 = \Big|\varepsilon\int\limits_{S_\varepsilon} q(\frac{x}{\varepsilon})u_0(x)\,v(x)\;ds - \frac{1}{|\Xi\cap\omega|}\int\limits_{\Omega_\varepsilon} < q >_S u_0(x)\,v(x)\,dx\Big| \\
\le C_2\,\varepsilon\,\|u_0\|_{H^1(\Omega^\varepsilon)}\|v\|_{H^1(\Omega^\varepsilon)}.
\end{aligned}$$

By means of the same arguments as those in the proof of Lemma 3 (see [5] and [9]) it is easy to show that

$$I_5 \le C_8\,\sqrt{\varepsilon}\;\|v\|_{H^1(\Omega^\varepsilon)},\quad I_6 \le C_9\,\sqrt{\varepsilon}\;\|v\|_{H^1(\Omega^\varepsilon)},\quad I_7 \le C_{10}\,\sqrt{\varepsilon}\;\|v\|_{H^1(\Omega^\varepsilon)}.$$

Clearly, the terms $I_1$ and $I_3$ admit the following bounds

$$|I_1| + |I_3| \leq C_{11}\varepsilon \|v\|_{H^1(\Omega^\varepsilon)}.$$

The identity $I_4 \equiv 0$ follows from properties of the functions $N_i(\xi),\ i = 1, \ldots, d$. Substituting $v = u_0 + \varepsilon u_1 - u_\varepsilon$ in (46) and taking into account all the previous estimates we arrive at (20). The theorem is proved. □

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# Homogenization of random non stationary parabolic operators

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## Introduction

We study homogenization problem for a random non stationary parabolic second order equation of the form

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla u^\varepsilon(x,t)\right) + \frac{1}{\varepsilon}g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})u^\varepsilon(x,t) + f(x,t), \quad (1)$$

with a small positive parameter $\varepsilon$. This model equation describes various processes in a medium with spatial microstructure whose characteristics are rapidly changing functions of time.

Throughout this article we assume that the spatial microstructure is periodic and that the characteristics of this microstructure are random stationary rapidly oscillating processes.

The presence in the equation of a large zero order term, linear or nonlinear, leads to rather unusual asymptotic behaviour of a solution of (1), as $\varepsilon$ tends to zero. We will show that almost sure (a.s.) homogenization result in general fails to hold and that a weaker averaging result takes place. Namely, under certain mixing conditions, a solution of (1) converges in law in a suitable functional space to a solution of a homogenized stochastic partial differential equation (SPDE).

Our aim is to justify this convergence and to investigate the properties of the limit SPDE.

The presence of a large factor in the lower order terms of the equation is natural when studying long term behaviour of solutions. We illustrate this with the following example. Many applications deal with parabolic operator of the form

$$\frac{\partial}{\partial t}v(y,s) = \operatorname{div}\left(a(y,s)\nabla v(y,s)\right) + \varepsilon g(y,s)v(y,s),$$

with a small potential $\varepsilon g(y,s)$, here $\varepsilon$ characterizes the range of oscillation of the potential. In order to study the behaviour of solutions at large time $s \sim \varepsilon^{-2}$, one can make the diffusive change of variables $x = \varepsilon y$, $t = \varepsilon^2 s$. In the new coordinates the equation reads

$$\frac{\partial}{\partial t} v = \operatorname{div}\left( a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) \nabla v \right) + \varepsilon^{-1} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) v,$$

it is similar to the equation (1).

First rigorous homogenization results for random elliptic and parabolic operators in divergence form were obtained in the works [5], [10]. After that this topic has been studied by many mathematicians, now it is well presented in the existing literature. However, some important problems in the field remain open.

It is known that, in contrast with periodic case, the presence of lower order terms in the equation with random coefficients might change crucially the effective behaviour of solutions.

In this work we consider an intermediate case of equations with lower order terms whose coefficients are periodic in spatial variables and random in time. Averaging problems for these equations with diffusive driving process were studied in [2] in linear case, and in [12] in nonlinear case. The convection diffusion problem of this type with a generic stationary driving process having good mixing properties, have been considered in [6].

## 1. The setup

This section is devoted to homogenization of equations of the form

$$\frac{\partial}{\partial t} u^\varepsilon(x,t) = \operatorname{div}\left( a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) \nabla u^\varepsilon(x,t) \right) + \frac{1}{\varepsilon} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) u^\varepsilon(x,t) + f(x,t), \qquad (2)$$

with generic random stationary in time and periodic in spatial variables coefficients. For this equation we consider a Cauchy problem in $\mathbb{R}^n \times (0,T)$ with the initial condition

$$u^\varepsilon(x,0) = u_0(x). \qquad (3)$$

Here and later on we assume that $u_0 \in L^2(\mathbb{R}^n)$ and $f \in L^2((0,T) \times \mathbb{R}^n)$.

*Remark 1.* The Cauchy problem has been chosen for the sake of definiteness. Initial boundary problems with Dirichlet or Neumann conditions can be studied in a similar way.

Problem (2)–(3) will be investigated under the following assumptions on the coefficients.

**H1.** The coefficients $a_{ij}(y,s)$ and $g(y,s)$ are $[0,1]^n$ periodic in $y$.

**H2.** The functions $a_{ij}(y,s)$ and $g(y,s)$ are stationary random functions of $s$ defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, with values in the space of periodic functions of $y$. We assume that $a_{ij}(y,s) = a_{ij}(y,s,\omega)$ and $g(y,s) = g(y,s,\omega)$ are measurable with respect to the $\sigma$-algebra $\mathcal{B}(\mathbb{T}^n) \times \mathcal{B}(-\infty,+\infty) \times \mathcal{F}$, where the symbol $\mathcal{B}$ stands for the Borel $\sigma$-algebra.

For simplicity we assume that $\Omega$ is equipped with a random dynamical system $T_t$ and that

$$a_{ij}(y,s,\omega) = \breve{a}_{ij}(y,T_s\omega), \quad g(y,s,\omega) = \breve{g}(y,T_s\omega), \tag{4}$$

where $\breve{a}_{ij}(y,\omega)$ and $\breve{g}(y,\omega)$ are given random function with values in $L^\infty(\mathbb{T}^n)$.

Let us recall that $T_t$ is a group of measurable transformations $T_t : \Omega \longrightarrow \Omega$ such that

- $T_{s_1}T_{s_2} = T_{s_1+s_2}, \quad T_0 = \mathrm{Id}$;
- $T_s$ preserves measure $\mathbf{P}$ for any $s \in \mathbb{R}$, i.e. $\mathbf{P}(T_s(\mathcal{G})) = \mathbf{P}(\mathcal{G})$ for any $\mathcal{G} \in \mathcal{F}$;
- $T_s(\omega)$ is a measurable map from $(\Omega\times\mathbb{R},\mathcal{F}\times\mathcal{B})$ to $(\Omega,\mathcal{F})$, where $\mathcal{B}$ stands for a Borel $\sigma$-algebra.

**H3.** Uniform ellipticity:

$$a_{ij}(y,\tau)\zeta_i\zeta_j \geq \lambda|\zeta|^2, \quad \lambda > 0,$$
$$|a_{ij}(y,\tau)| \leq \lambda^{-1}, \quad |g(y,s)| \leq \lambda^{-1}$$

for all $y$, $\tau$ and $\zeta \in \mathbb{R}^n$.

**H4.** Centering condition. The average of $g(z,s)$ is equal to zero that is

$$\mathbf{E}\int\limits_{[0,1]^n} g(z,s)dz = 0 \tag{5}$$

for all $s \in \mathbb{R}$.

In order to formulate one more assumption we first recall the definition of mixing coefficients.

Let $\xi_s$ be a stationary random process defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, and denote $\mathcal{F}_{\leq t} = \sigma\{\xi_s,\ s \leq t\}$ and $\mathcal{F}_{\geq t} = \sigma\{\xi_s,\ s \geq t\}$.

The function $\kappa(\gamma)$ defined by

$$\kappa(\gamma) = \sup_{E_1\in\mathcal{F}_{\leq t},\, E_2\in\mathcal{F}_{\geq(t+\gamma)}} |\mathbf{P}(E_1)\mathbf{P}(E_2) - \mathbf{P}(E_1\cap E_2)|,$$

is called strong mixing coefficient of the process $\xi_\cdot$. Notice that since $\xi_\cdot$ is stationary, $\kappa(\gamma)$ does not depend on $t$.

The function $\varphi(\gamma)$ defined by

$$\varphi(\gamma) = \sup_{\substack{E_1 \in \mathcal{F}_{\leq t},\ E_2 \in \mathcal{F}_{\geq(t+\gamma)} \\ \mathbf{P}(E_2) \neq 0}} \left|\mathbf{P}(E_1) - \frac{\mathbf{P}(E_1\cap E_2)}{\mathbf{P}(E_2)}\right|,$$

is called the *uniform mixing coefficient* of $\xi_\cdot$.

The function $\rho(\gamma)$ defined by

$$\rho(\gamma) = \sup_{\eta_1,\,\eta_2} \left| \frac{\mathbf{E}((\eta_1 - \mathbf{E}\eta_1)(\eta_2 - \mathbf{E}\eta_2))}{\sqrt{\mathbf{E}\eta_1^2\,\mathbf{E}\eta_2^2}} \right|,$$

$$\eta_1 \in L^2(\Omega, \mathcal{F}_{\le t}, \mathbf{P}),\ \eta_2 \in L^2(\Omega, \mathcal{F}_{\ge(t+\gamma)}, \mathbf{P}).$$

is called the *maximum correlation coefficient* of $\xi_\cdot$.

We now consider $\sigma$-algebras $\mathcal{F}_{\le t}$ and $\mathcal{F}_{\ge t}$, generated by the coefficients $a(y,t), g(y,t)$ of operator (2), and impose the following condition on the corresponding mixing coefficients

**H5.** At least one of the following conditions holds true.

$$\int\limits_0^\infty \sqrt{\kappa(\gamma)}d\gamma < \infty, \qquad \int\limits_0^\infty \varphi(\gamma)d\gamma < \infty, \qquad \int\limits_0^\infty \rho(\gamma)d\gamma < \infty$$

*Remark 2.* Condition **H4** can be assumed without loss of generality. Indeed, the relation (5) can be achieved by means of the following factorization of unknown function in (2)

$$\tilde{u}^\varepsilon(x,t) = \exp(<\bar{g}>/\varepsilon)u(x,t),$$

with

$$<\bar{g}> = \mathbf{E}\int\limits_{[0,1]^n} g(z,s)dz.$$

If $<\bar{g}>\ne 0$, then the homogenization takes place on the background of exponential growth or decay of the solution.

Under conditions **H1-H3** problem (2)-(3) is well posed for each $\varepsilon > 0$.

**Lemma 1.** *Let* **H1**-**H3** *be fulfilled. Then for each $\varepsilon > 0$ problem (2)-(3) has a unique solution $u^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^n)) \cap C(0,T;L^2(\mathbb{R}^n))$ for all $\omega \in \Omega$. This solution defines a measurable mapping*

$$u^\varepsilon\ :\ (\Omega, \mathcal{F}) \longrightarrow (L^2(0,T;H^1(\mathbb{R}^n)) \cap C(0,T;L^2(\mathbb{R}^n)), \mathcal{B}).$$

*The estimate holds*

$$\begin{aligned} &\|u^\varepsilon\|_{C((0,T);L^2(\mathbb{R}^n))} + \|u^\varepsilon\|_{L^2((0,T);H^1(\mathbb{R}^n))} \\ &\le C(\varepsilon)(\|f\|_{L^2(0,T;H^{-1}(\mathbb{R}^n)} + \|u_0\|_{L^2(\mathbb{R}^n)}). \end{aligned} \tag{6}$$

*Proof.* The existence and the uniqueness of a solution as well as a priori estimate (6) are standard. The measurability is the consequence of the fact that $u^\varepsilon$ depends continuously on the data of the problem. □

## 2. Factorization of the equation

It is convenient to represent $g(y,s)$ as a sum

$$g(y,t) =< g > (t) + \tilde{g}(y,t), \qquad < g > (t) \stackrel{\text{def}}{=} \int\limits_{[0,1]^n} g(z,t)dz,$$

and to introduce a new unknown function $v^\varepsilon = v^\varepsilon(x,t)$ as follows

$$u^\varepsilon(x,t) = v^\varepsilon(x,t) \exp\left(\frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds\right), \tag{7}$$

It is straightforward to check that $v^\varepsilon$ satisfies the equation

$$\frac{\partial}{\partial t}v^\varepsilon(x,t) = \operatorname{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla v^\varepsilon(x,t)\right) + \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^\varepsilon(x,t)$$

$$+f(x,t)\exp\left(-\frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds\right), \tag{8}$$

$$v^\varepsilon(x,0) = u_0(x).$$

This problem will be studied in the following sections. In the remaining part of this section we deal with the exponential factor in (7).

**Lemma 2.** *Suppose that at least one of the conditions* **H5** *holds. Then the process*

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon}\int\limits_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds \tag{9}$$

*satisfies functional Central Limit Theorem (invariance principle) with zero mean and the diffusion given by*

$$\sigma^2 = 2\int\limits_0^\infty \mathbf{E} < g > (0) < g > (s)ds. \tag{10}$$

*That is the process* $\{\zeta_t^\varepsilon\}$ *converges in law in the space* $C[0,\infty)$ *to the process* $\{\sigma W_t\}$*, where* $W_t$ *is a standard Brownian motion.*

The **proof** of this statement can be found for instance in [9], Chapter 9.
As a consequence of the lemma we obtain the convergence

$$\exp\left(\frac{1}{\varepsilon}\int_0^t < g > \left(\frac{s}{\varepsilon^2}\right)ds\right) \xrightarrow{\mathcal{L}} \exp(\sigma W_t) \tag{11}$$

in the space $C[0,\infty)$.

## 3. A priori estimates for the factorized equation

In this section we derive a priori estimates for a solution of problem (8) and of more general Cauchy problem of the form

$$\frac{\partial}{\partial t} z^\varepsilon(x,t) = \operatorname{div}\left( a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) \nabla z^\varepsilon(x,t) \right) + \frac{1}{\varepsilon} \tilde{g}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) z^\varepsilon(x,t) + h(x,t), \tag{12}$$

$$z^\varepsilon(x,0) = z_0(x),$$

which involves a nontrivial right hand side.

**Proposition 1.** *A solution $z^\varepsilon$ of problem (12) admits an estimate*

$$\|z^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|z^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))} \le C(\|z_0\|_{L^2(\mathbb{R}^n)} + \|h\|_{L^2(0,T;H^{-1}(\mathbb{R}^n))}) \tag{13}$$

*with a constant $C$ which does not depend on $\varepsilon$.*

*Proof.* By construction the function $\tilde{g}(y,s)$ has zero average in variable $y$ for all $s$ and $\omega$. Therefore, the equation $\Delta \tilde{Q} = \tilde{g}$ is solvable in the space of periodic functions. Denote $\tilde{q} = \nabla_y \tilde{Q}$. Since $\|\tilde{g}\|_{L^\infty} < \infty$, we have $|\tilde{q}(y,s)| \le C$.

Clearly, $q(y,s)$ satisfies the relation $\mathrm{div_y} \tilde{q}(y,s) = \tilde{g}(y,s)$. In coordinates $x = \varepsilon y$, $t = \varepsilon^2 s$ it reads

$$\varepsilon \mathrm{div_x} \tilde{q}^\varepsilon(x,t) = \tilde{g}^\varepsilon(x,t); \tag{14}$$

Here and afterwards for a generic function $F(y,s)$ we use the notation

$$F_\varepsilon(x,t) = F(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}), \quad \frac{\partial}{\partial y_i} F_\varepsilon(x,t) = \frac{\partial}{\partial y_i} F(y, \frac{t}{\varepsilon^2})\Big|_{y=\frac{x}{\varepsilon}}, \tag{15}$$

$$\frac{\partial}{\partial s} F_\varepsilon(x,t) = \frac{\partial}{\partial s} F(\frac{x}{\varepsilon}, s)\Big|_{s=\frac{t}{\varepsilon^2}}$$

Multiplying the equation (12) by $z^\varepsilon$ and integrating the resulting relation over the set $\mathbb{R}^n \times (0,T)$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} (z^\varepsilon(x,t))^2 dx - \int_{\mathbb{R}^n} (z_0(x))^2 dx \\
&= -\int_0^t \int_{\mathbb{R}^n} a_{ij,\varepsilon}(x,\tau) \nabla z^\varepsilon(x,\tau) \cdot \nabla z^\varepsilon(x,\tau) dx d\tau + \\
&\quad + \varepsilon^{-1} \int_0^t \int_{\mathbb{R}^n} \tilde{g}_\varepsilon(x,\tau) (z^\varepsilon(x,\tau))^2 dx d\tau + \int_0^t \int_{\mathbb{R}^n} z^\varepsilon(x,\tau) h(x,\tau) dx d\tau.
\end{aligned}$$

Considering (14), after multiple integration by parts we get

$$\int_{\mathbb{R}^n} (z^\varepsilon(x,t))^2 dx + \int_0^t \int_{\mathbb{R}^n} a_{ij,\varepsilon}(x,\tau)\nabla z^\varepsilon(x,\tau)\cdot\nabla z^\varepsilon(x,\tau)dxd\tau$$
$$= \int_{\mathbb{R}^n} (z_0(x))^2 dx + \int_0^t \int_{\mathbb{R}^n} z^\varepsilon(x,\tau)\tilde{q}_\varepsilon(x,\tau)\cdot\nabla z^\varepsilon(x,\tau)dxd\tau +$$
$$+ \int_0^t \int_{\mathbb{R}^n} z^\varepsilon(x,\tau)h(x,\tau)dxd\tau. \tag{16}$$

Denote the right hand side here by $R^\varepsilon(t)$. For each $\gamma > 0$ we have

$$|R^\varepsilon(t)| \le \int_{\mathbb{R}^n} (z_0(x))^2 dx + \gamma^{-1} \int_0^t \int_{\mathbb{R}^n} |z^\varepsilon(x,\tau)\tilde{q}_\varepsilon(x,\tau)|^2 dxd\tau$$
$$+\gamma \int_0^t \int_{\mathbb{R}^n} |\nabla z^\varepsilon(x,\tau)|^2 dxd\tau + \int_0^t \|z^\varepsilon(\cdot,\tau)\|_{H^1(\mathbb{R}^n)}\|h(\cdot,\tau)\|_{H^{-1}(\mathbb{R}^n)}d\tau$$
$$\le \int_{\mathbb{R}^n} (z_0(x))^2 dx + C\gamma^{-1} \int_0^t \int_{\mathbb{R}^n} |z^\varepsilon(x,\tau)|^2 dxd\tau + \gamma \int_0^t \int_{\mathbb{R}^n} |\nabla z^\varepsilon(x,\tau)|^2 dxd\tau$$
$$+\gamma^{-1} \int_0^t \|h(\cdot,\tau)\|^2_{H^{-1}(\mathbb{R}^n)}d\tau + \gamma \int_0^t \int_{\mathbb{R}^n} (|z^\varepsilon(\cdot,\tau)|^2 + |\nabla z^\varepsilon(\cdot,\tau)|^2)dxd\tau.$$

It remains to combine this bound with (16). The desired estimate (13) now follows from Gronwall lemma. □

## 4. Auxiliary problems

Passage to the limit in problem (8) requires introducing a number of auxiliary functions usually called *correctors*. These correctors will be defined as solutions of auxiliary parabolic equations. This section is devoted to those auxiliary problems and their properties.

Denote $A = \frac{\partial}{\partial y_i} a_{ij}(y,s)\frac{\partial}{\partial y_j}$ and consider in a cylinder $\mathbb{T}^n \times (-\infty, +\infty)$ the following two equations:

$$\frac{\partial}{\partial s}\chi^j(y,s) - A\chi^j(y,s) = \frac{\partial}{\partial y_i} a_{ij}(y,s) \tag{17}$$

$$\frac{\partial}{\partial s}G(y,s) - AG(y,s) = \tilde{g}(y,s) \tag{18}$$

**Proposition 2.** *Equations (17) and (18) have stationary solutions in the space* $L^\infty(-\infty,+\infty; C(\mathbb{T}^n)) \cap L^2_{\rm loc}(-\infty,+\infty; H^1(\mathbb{T}^n))$*. Each of these solutions is unique up to a (random) additive constant. The estimates*

$$\|\chi\|_{L^2(N,N+1;H^1(\mathbb{T}^n))} \le C, \qquad \|G\|_{L^2(N,N+1;H^1(\mathbb{T}^n))} \le C, \tag{19}$$

*hold uniformly in* $N \in \mathbb{R}$*. Moreover, the constant* $C$ *is deterministic and only depends on* $\lambda$ *in* **H3**.

*Proof.* Let us show that (17) has a stationary solution. To this end we consider the following Cauchy problem

$$\frac{\partial}{\partial s}\chi_N^j(y,s) - A\chi_N^j(y,s) = \frac{\partial}{\partial y_i}a_{ij}(y,s)\mathbf{1}_{[N,N+1)}(s), \quad (y,s)\in\mathbb{T}^n\times(N,+\infty)$$

$$\chi_N^j(y,N) = 0. \tag{20}$$

For $s < N$ we set $\chi_N^j(y,s) = 0$. By the Nash estimates (see [8]), the function $\chi_N^j$ is continuous and satisfies the upper bound

$$\|\chi_N^j\|_{L^\infty((-\infty,N+1]\times\mathbb{T}^n)} \le c_1(\gamma).$$

Since the right hand side in (20) is equal to zero for all $s > N+1$, by the maximum principle the last estimate is valid for all $s$:

$$\|\chi_N^j\|_{L^\infty((-\infty,+\infty)\times\mathbb{T}^n)} \le c_1(\gamma).$$

We want to show that $\chi_N^j(y,s)$ decays exponentially as $(s-N)\to\infty$.

**Lemma 3.** *There are nonrandom independent of $N$ constants $c_2(\lambda) > 0$ and $c_3(\lambda) > 0$ such that*

$$|\chi_N^j(y,s)| \le c_2\exp(-c_3(s-N)) \tag{21}$$

*Proof* of the lemma. Notice that

$$\int_{\mathbb{T}^n}\chi_N^j(y,s)dy = 0 \tag{22}$$

for each $s\in\mathbb{R}$. Indeed, integrating the equation (17) on the cylinder $(s_1,s_2)\times\mathbb{T}^n$, one has $\int_{\mathbb{T}^n}\chi_N^j(y,s_1)dy = \int_{\mathbb{T}^n}\chi_N^j(y,s_2)dy$. Then (22) follows from the equality $\chi_N^j(y,N) = 0$.

By the Poincaré inequality, considering (22) and **H3**, we get for each $s$

$$\int_{\mathbb{T}^n} a_{ij}(y,s)\frac{\partial}{\partial y_i}\chi_N^k(y,s)\frac{\partial}{\partial y_j}\chi_N^k(y,s)dy \ge \tag{23}$$

$$\ge \lambda^{-1}\int_{\mathbb{T}^n}|\nabla_y\chi_N^k(y,s)|^2dy \ge c_4\lambda^{-1}\int_{\mathbb{T}^n}|\chi_N^k(y,s)|^2dy.$$

Now we multiply (17) by $\chi_N^j(y,s)$ and integrate the resulting relation on the cylinder $(\mathbb{T}^n\times(s_1,s_2)$. This gives

$$\|\chi_N^k(\cdot,s_2)\|^2_{L^2(\mathbb{T}^n)} - \|\chi_N^k(\cdot,s_1)\|^2_{L^2(\mathbb{T}^n)} \le -c_4\lambda^{-1}\int_{s_1}^{s_2}\|\chi_N^k(\cdot,s)\|^2_{L^2(\mathbb{T}^n)}ds \tag{24}$$

for all $s_1$ and $s_2$ such that $N+1\le s_1\le s_2$. This implies the bound

$$\|\chi_N^k(\cdot,s)\|^2_{L^2(\mathbb{T}^n)} \le c_2 \exp(-c_3(s-(N+1))$$

It is also clear that the function $\|\chi_N^k(\cdot,s)\|^2_{L^2(\mathbb{T}^n)}$ is monotone on the interval $(N+1,\infty)$. To complete the proof of the lemma it remains to apply once again the Nash inequality. □

We define a vector function $\chi(y,s)$ by

$$\chi^k(y,s) = \sum_{N=-\infty}^{+\infty} \chi_N^k(y,s) \tag{25}$$

By construction and in view of the last Lemma, $\chi^j(y,s)$ solves the equation (17) and satisfies the estimate (19). We want to show that $\chi^j(y,s)$ is stationary. The fact that any finite dimensional distribution of this random function is invariant with respect to all integer shifts easily follows from the stationarity of $a_{ij}(\cdot,s)$.

Taking in the above procedure an arbitrary rational step size $q$ instead of 1, we construct a solution of equation (17) whose finite dimensional distributions are invariant with respect to any shift of the form $kq$ with integer $k$. It is easy to check that this new solution coincides with $\chi^j(y,s)$. Thus, by arbitrariness of $q$, the finite dimensional distributions of $\chi^j(y,s)$ are invariant with respect to any rational shift. Now the stationarity of $\chi^j(y,s)$ follows by continuity arguments.

The uniqueness of a stationary solution up to a (random) additive constant follows from Lemma 3. Indeed, if we assume the existence of two distinct stationary solutions with zero average, then their difference vanishes as $s\to\infty$. This contradicts the stationarity. □

We impose the following normalization conditions for $\chi^k(y,s)$ and $G(y,s)$:

$$\int_{\mathbb{T}^n} \chi^j(y,s)dy = 0, \qquad \int_{\mathbb{T}^n} G(y,s)dy = 0 \tag{26}$$

This makes the choice of the corresponding additive constants unique.
We set

$$\tilde\chi^k(y,\omega) = \chi^k(y,0,\omega), \quad \tilde G(y,\omega) = G(y,0,\omega).$$

Since $\chi^k(y,s)$ and $G(y,s)$ are continuous in $s$, the functions $\tilde\chi^k$ and $G$ are well defined. By definition $a_{ij}(y,s+\tau,\omega) = a_{ij}(y,s,T_\tau\omega)$. Therefore, considering the uniqueness of solution of problem (17), we have $\chi^k(y,s+\tau,\omega) = \chi^k(y,s,T_\tau\omega)$. In particular,

$$\chi^k(y,s,\omega) = \chi^k(y,0,T_s\omega) = \tilde\chi^k(y,T_s\omega).$$

Similarly,

$$G(y,s,\omega) = G(y,0,T_s\omega) = \tilde G(y,T_s\omega). \qquad \square$$

## 5. Homogenization of the factorized equation

We begin by considering the equation (2) in the particular case $f = 0$. Our aim is to show that in this case factorized problem (8) admits a.s. homogenization.

**Theorem 1.** *Let $f = 0$. Then under our standing assumptions a solution $v^\varepsilon$ of problem (8) converges a.s., as $\varepsilon \to 0$, in the space $L^\infty(0,T;L^2(\mathbb{R}^n))$ towards a solution of the following Cauchy problem*

$$\begin{aligned} \frac{\partial}{\partial t} v^0(x,t) &= \mathrm{div}\,\big(\hat{a}\nabla v^0(x,t)\big) + \hat{b}\cdot\nabla v^0(x,t) + \hat{G}v^0(x,t),\\ v^0(x,0) &= u_0(x). \end{aligned} \tag{27}$$

*The homogenized equation has constant coefficients defined by*

$$\hat{a}_{ij} = \mathbf{E}\int\limits_{\mathbb{T}^n} a_{ik}(y,s)\big(\delta_{kj} + \frac{\partial}{\partial y_k}\chi^j(y,s)\big)dy, \tag{28}$$

$$\hat{b}_i = \mathbf{E}\int\limits_{\mathbb{T}^n} \Big(\tilde{g}(y,s)\chi^i(y,s) + a_{ij}\frac{\partial}{\partial y_j}G(y,s)\Big)dy, \tag{29}$$

$$\hat{G} = \mathbf{E}\int\limits_{\mathbb{T}^n} \tilde{g}(y,s)G(y,s)dy. \tag{30}$$

*Proof.* Assume for a while that $u_0 \in C_0^\infty$. Then a solution $v^0$ of problem (27) is a $C^\infty$ function which vanishes at infinity, as well as its partial derivatives, faster than any negative power of $(1+|x|)$. We then substitute the following *ansatz*

$$\check{v}^\varepsilon(x,t) = v^0(x,t) + \varepsilon\chi(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\cdot\nabla v^0 + \varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$

in the equation (8). Considering (17) and (18), after straightforward rearrangements we get

$$\frac{\partial}{\partial t}(\check{v}^\varepsilon(x,t) - v^\varepsilon(x,t)) - \mathrm{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(\check{v}^\varepsilon(x,t) - v^\varepsilon(x,t))\right)$$

$$-\frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})(\check{v}^\varepsilon(x,t) - v^\varepsilon(x,t))$$

$$= \frac{\partial}{\partial t}v^0 + \frac{1}{\varepsilon}\frac{\partial}{\partial s}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0 + \frac{1}{\varepsilon}\frac{\partial}{\partial s}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0 + \varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0$$

$$+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0 - a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0 - \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_j}v^0 - \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$

$$-\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_j}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0 - a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_k\partial x_j}v^0$$

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial^2}{\partial x_k\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0$$
$$-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0-\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_j}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$
$$-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_j}v^0-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial}{\partial x_j}v^0$$
$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$
$$=\frac{\partial}{\partial t}v^0-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_k\partial x_j}v^0$$
$$-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial^2}{\partial x_k\partial x_j}v^0-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_k}v^0$$
$$-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_j}v^0-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\frac{\partial}{\partial x_j}v^0$$
$$-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0+\varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0$$
$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0.$$

Substituting the right hand side of the equation (27) in place of $\frac{\partial}{\partial t}v^0$ gives

$$\frac{\partial}{\partial t}(\check{v}^\varepsilon(x,t)-v^\varepsilon(x,t))-\mathrm{div}\left(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\nabla(\check{v}^\varepsilon(x,t)-v^\varepsilon(x,t))\right)$$
$$-\frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})(\check{v}^\varepsilon(x,t)-v^\varepsilon(x,t))$$
$$=\left\{\hat{a}_{ij}-a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})-a_{ik}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_k}\chi^j(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})-\frac{\partial}{\partial y_k}\left(a_{kj}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^i(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\right\}$$
$$\times\frac{\partial^2}{\partial x_i\partial x_j}v^0+\left\{\hat{b}_i-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^j(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})-a_{ji}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial y_i}G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right.$$
$$\left.-\frac{\partial}{\partial y_i}\left(a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right)\right\}\frac{\partial}{\partial x_j}v^0+\left\{\hat{G}-\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\right\}v^0$$
$$+\varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0$$
$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0.$$

Denote the right hand side of the last formula by $R_2^\varepsilon$. Since the expressions in the figure brackets are periodic in spatial variables and have zero average for any $t$, they can be represented as in (14):

$$\Big\{\hat a_{ij}-a_{ij}(y,s)-a_{ik}(y,s)\frac{\partial}{\partial y_k}\chi^j(y,s)$$
$$-\frac{\partial}{\partial y_k}\big(a_{kj}(y,s)\chi^i(y,s)\big)\Big\}=\mathrm{div}_y\kappa_{1,ij}(y,s)$$
$$\Big\{\hat b_i-\tilde g(y,s)\chi^j(y,s)-a_{ji}(y,s)\frac{\partial}{\partial y_i}G(y,s)$$
$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y,s)G(y,s)\right)\Big\}=\mathrm{div}_y\kappa_{2,i}(y,s)$$
$$\Big\{\hat G-\tilde g(y,s)G(y,s)\Big\}=\mathrm{div}_y\kappa_3(y,s)$$

where the functions $\kappa_1,ij(y,s)$, $\kappa_2,i(y,s)$ and $\kappa_3(y,s)$ are periodic in $y$ and satisfy the estimates

$$\|\kappa_{1,ij}\|_{L^2((s,s+1)\times\mathbb{T}^n)}\le C,\ \|\kappa_{2,i}\|_{L^2((s,s+1)\times\mathbb{T}^n)}\le C,\ \|\kappa_3\|_{L^2((s,s+1)\times\mathbb{T}^n)}\le C,$$

uniformly in $s\in\mathbb{R}$. Then $R_2^\varepsilon$ takes the form

$$R_2^\varepsilon=\varepsilon\mathrm{div}_x\kappa_{1,ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0+\varepsilon\mathrm{div}_x\kappa_{2,i}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial x_i}v^0+\varepsilon\mathrm{div}_x\kappa_3(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})v^0$$
$$+\varepsilon\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}\frac{\partial}{\partial x_k}v^0+\varepsilon G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial}{\partial t}v^0 \tag{31}$$
$$-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\chi^k(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^3}{\partial x_k\partial x_i\partial x_j}v^0-\varepsilon a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})G(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})\frac{\partial^2}{\partial x_i\partial x_j}v^0.$$

Due to the properties of $v^0$ this implies the estimate

$$\|R_2^\varepsilon\|_{L^2((0,T);H^{-1}(\mathbb{R}^n))}\le C\varepsilon.$$

Therefore, by Proposition 1

$$\|\check v^\varepsilon-v^\varepsilon\|_{L^2((0,T);H^1(\mathbb{R}^n))}\le C\varepsilon,\quad \|\check v^\varepsilon-v^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^n))}\le C\varepsilon. \tag{32}$$

Combining the latter estimate with an evident bound

$$\|\check v^\varepsilon-v^0\|_{L^\infty(\mathbb{R}^n\times(0,T))}\le C\varepsilon$$

we obtain the desired statement for all smooth $u_0$ with compact support.

In order to prove this result for general $u_0\in L^2(\mathbb{R}^n)$ we introduce a family of functions $u_0^\delta\in C_0^\infty(\mathbb{R}^n)$ such that $\|u_0^\delta-u_0\|_{L^2(\mathbb{R}^n)}\le\delta$. If we denote $v^{\delta,\varepsilon}$ a solution of problem (8) with initial condition $u_0^\delta$, then according to Proposition 1 the estimate

$$\|v^{\delta,\varepsilon}-v^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\le C\delta$$

holds. Evidently, we have $\|v^{\delta,0}-v^0\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\le C\delta$. As was proved above, $v^{\delta,\varepsilon}$ converges, as $\varepsilon\to0$, to $v^{\delta,0}$ in $L^\infty(0,T;L^2(\mathbb{R}^n))$. Therefore,

$$\limsup_{\varepsilon\to0}\|v^\varepsilon-v^0\|_{L^\infty(0,T;L^2(\mathbb{R}^n))}\le C\delta.$$

Since $\delta$ is an arbitrary positive number, the result follows. □

*Remark 3.* In the proof of Theorem 1 we did not use the mixing conditions **H5**, but only the ergodicity of the dynamical system $T_s$. The statement of this theorem remains valid for any ergodic dynamical system $T_s$ without mixing assumptions.

In particular, we obtain the following result.

**Corollary 1.** *Let the coefficients of problem (2)-(3) be periodic in spatial variables and stationary ergodic in time, and suppose the uniform ellipticity conditions* **H3**. *Assume, furthermore, that* $\int\limits_{\mathbb{T}^n} \tilde{g}(y,s)dy = 0$ *for all* $s \in \mathbb{R}$ *a.s. Then problem (2)-(3) admits a.s. homogenization and the limit operator is a non random parabolic operator with constant coefficients given by (28)-(30).*

It should be noted that the methods developed in the proof of Theorem 1 apply to the equation (12) with a right hand side $h(x,t) \in L^2((0,T)\times\mathbb{R}^n)$. The following result holds true.

**Theorem 2.** *Let* $h(x,t) \in L^2((0,T)\times\mathbb{R}^n)$ *and* $z_0 \in L^2(\mathbb{R}^n)$. *Then a solution of problem (12) converges a.s., as* $\varepsilon \to 0$, *in the space* $L^2(0,T;H^1(\mathbb{R}^n))$ *towards a solution of problem*

$$\frac{\partial}{\partial t} z^0(x,t) = \operatorname{div}\big(\hat{a}\nabla z^0(x,t)\big) + \hat{b}\cdot\nabla z^0(x,t) + \hat{G}z^0(x,t) + h(x,t),$$
$$z^0(x,0) = z_0(x), \tag{33}$$

*with constant non random coefficients given by (28)–(30).*

We proceed by studying problem (8) with non trivial right hand side. We denote

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon}\int_0^t < g > \Big(\frac{s}{\varepsilon^2}\Big)ds.$$

and introduce $V^\varepsilon(x,t)$ to be a solution to the following Cauchy problem in $\mathbb{R}^n \times (0,T)$

$$\frac{\partial}{\partial t} V^\varepsilon(x,t) = \operatorname{div}\left(\hat{a}\nabla V^\varepsilon(x,t)\right) + \hat{b}\cdot\nabla V^\varepsilon(x,t) + \hat{G}V^\varepsilon(x,t) + f(x,t)\exp(-\zeta_t^\varepsilon),$$
$$V^\varepsilon(x,0) = u_0(x), \tag{34}$$

with coefficients defined in (28)–(30). It is convenient to represent a solution $v^\varepsilon$ of problem (8) as a sum $v^\varepsilon = V^\varepsilon + (v^\varepsilon - V^\varepsilon)$. We will show that $(v^\varepsilon - V^\varepsilon)$ tends to zero in probability, as $\varepsilon \to 0$, in the norm of $L^2((0,T)\times\mathbb{R}^n)$, while $V^\varepsilon$ converges in law in $L^2((0,T)\times\mathbb{R}^n)$ to a solution of Cauchy problem

$$\frac{\partial}{\partial t} v^0(x,t) = \operatorname{div}\big(\hat{a}\nabla v^0(x,t)\big) + \hat{b}\cdot\nabla v^0(x,t) + \hat{G}v^0(x,t) + f(x,t)\exp(-\sigma W_t),$$
$$v^0(x,0) = u_0(x), \tag{35}$$

with $\sigma$ given by (10). Notice that this equation has a random right hand side.

**Proposition 3.** *The $L^2((0,T)\times\mathbb{R}^n)$ norm of the difference $(v^\varepsilon - V^\varepsilon)$ tends to zero in probability as $\varepsilon \to 0$.*

*Proof.* By Lemma 2 the process $\zeta_t^\varepsilon$ converges in law in $C(0,T)$, as $\varepsilon \to 0$, towards $\sigma W_t$. Therefore, a random function $f(x,t)\zeta_t^\varepsilon$ converges in law in $L^2((0,T)\times\mathbb{R}^n)$ to $\sigma f(x,t)W_t$. By the Prokhorov theorem this implies that for any $\delta > 0$ there is a compact set $K^\delta \subset L^2((0,T)\times\mathbb{R}^n)$ such that $\mathbf{P}\{f(x,t)\zeta_t^\varepsilon \notin K^\delta\} \le \delta$. Consider a finite $\delta$-net in $K^\delta$, for which we use the notation $\{h_j\}$, $j = 1, 2, \ldots, N(\delta)$, and denote $z_j^\varepsilon(x,t)$ and $z_j^0(x,t)$ respectively solutions of problem (12) and (33) with right hand side $h_j(x,t)$ and initial condition $u_0(x)$.

By Theorem 1 for any $\delta > 0$ there exists $\varepsilon_0(\delta) > 0$ such that

$$\max \mathbf{P}\{\|z_j^\varepsilon - z_j^0\|_{L^2} > \delta\} < \frac{\delta}{N(\delta)}$$

Let $\mathcal{E}_j$ be the following events

$$\mathcal{E}_j = \{\Omega\,:\, \|f(x,t)\zeta_t^\varepsilon - h_j(x,t)\|_{L^2} < \delta\}.$$

By construction $\mathbf{P}\big(\Omega \setminus \bigcup_{j=1}^{N(\delta)} \mathcal{E}_j\big) < \delta$. Considering the estimate (13) and similar estimate for the homogenized problem, we conclude that for all $\omega \in \mathcal{E}_j$ the inequality holds

$$\|V^\varepsilon - v^\varepsilon\|_{L^2} \le \|V^\varepsilon - z_j^0\|_{L^2} + \|z_j^0 - z_j^\varepsilon\|_{L^2} + \|z_j^\varepsilon - v^\varepsilon\|_{L^2}$$
$$\le 2C\delta + \|z_j^0 - z_j^\varepsilon\|_{L^2},$$

with a constant $C$ that depends neither on $\varepsilon$ nor on $\omega$. Thus

$$\mathbf{P}\{\|V^\varepsilon - v^\varepsilon\|_{L^2} > (2C+1)\delta\}$$
$$\le \mathbf{P}\Big(\Omega \setminus \bigcup_{j=1}^{N(\delta)} \mathcal{E}_j\Big) + \sum_{j=1}^{N(\delta)} \mathbf{P}\big\{\mathcal{E}_j \cap \big(\|V^\varepsilon - v^\varepsilon\|_{L^2} > (2C+1)\delta\big)\big\}$$
$$\delta + \sum_{j=1}^{N(\delta)} \mathbf{P}\{\|z_j^0 - z_j^\varepsilon\|_{L^2} > \delta\} \le \delta + \sum_{j=1}^{N(\delta)} \frac{\delta}{N(\delta)} = 2\delta.$$

This implies the required convergence in probability. □

**Proposition 4.** *The function $V^\varepsilon$ converges in law, as $\varepsilon \to 0$, in $L^2((0,T)\times\mathbb{R}^n)$ towards a solution $v^0$ of problem (35).*

*Proof.* Notice that a solution of problem (33) as a functional of the right hand side defines a continuous mapping from $L^2((0,T)\times\mathbb{R}^n)$ to $L^2((0,T);H^1(\mathbb{R}^n))\cap L^\infty(0,T;L^2(\mathbb{R}^n))$. Then the convergence in law of the right hand side in (33) implies the convergence in law, in the corresponding functional space, of $V^\varepsilon$, and the desired statement follows. □

We summarize the above assertions in the following theorem.

**Theorem 3.** *Under conditions* **H1**–**H5** *the solution of factorized problem (8) converges in law, as* $\varepsilon \to 0$*, in the strong topology of the space* $L^2((0,T) \times L^2(\mathbb{R}^n))$ *towards a solution of problem (35) whose coefficients are given by (28)–(30).*

*Proof.* This statement is a consequence of Propositions 3 and 4. □

## 6. Homogenization of the original equation

We now turn to the homogenization of the original problem (2)–(3).

Notice first that the random process

$$(\exp(\zeta_t^\varepsilon), \exp(-\zeta_t^\varepsilon))$$

converges in law in $(C[0,T])^2$ to the process

$$(\exp(\sigma W_t), \exp(-\sigma W_t)),$$

where $\zeta_t^\varepsilon$ and $\sigma$ are defined in (9) and(10) respectively.

The asymptotic behaviour of a solution to problem (2)–(3) is described by the following

**Theorem 4.** *Let conditions* **H1**–**H5** *be fulfilled. Then, as* $\varepsilon \to 0$*, a solution* $u^\varepsilon$ *of problem (2)–(3) converges in law in the strong topology of* $L^2(\mathbb{R}^n \times (0,T))$ *to a solution of the following stochastic partial differential equation*

$$d\hat{u} = \Big(\hat{a}_{ij} \frac{\partial^2 \hat{u}}{\partial x_i \partial x_j} + \hat{b}_i \frac{\partial \hat{u}}{\partial x_i} + \hat{g}\hat{u}\Big) dt + \sigma \hat{u}\, dW_t + f(x,t), \tag{36}$$

$$\hat{u}(x,0) = u_0(x),$$

*with* $\hat{g} = \hat{G} + \frac{1}{2}\sigma^2$ *and* $\hat{a}_{ij}$*,* $\hat{b}_i$ *and* $\hat{G}$ *given by (28)–(30);* $\sigma$ *is defined in (10).*

According to [3] problem (36) is well posed and has a unique solution. Hence, the limit law is well defined.

*Remark 4.* If $\int_{\mathbb{T}^n} g(z,s)dz = 0$ for almost all $s$ then $\sigma$ is equal to zero and the limit problem (36)is deterministic. As was already mentioned, in this case $u^\varepsilon$ converges a.s.

*Proof (Theorem 4).* The solution $u^\varepsilon$ of problem (2)–(3) can be written as a sum

$$u^\varepsilon(x,t) = (v^\varepsilon(x,t) - V^\varepsilon(x,t)) \exp(\zeta_t^\varepsilon) + V^\varepsilon(x,t) \exp(\zeta_t^\varepsilon),$$

where $v^\varepsilon$ and $V^\varepsilon$ satisfies (8) and (34) respectively, and $\zeta_t^\varepsilon$ is defined in (9). By Proposition 3, the factor $(v^\varepsilon(x,t) - V^\varepsilon(x,t))$ in the first term on the right hand side converges in probability to zero in $L^2(\mathbb{R}^n \times (0,T))$ norm. Since by Lemma 2 the function $\zeta_t^\varepsilon$ converges in law in the space $C[0,T]$, the product $(v^\varepsilon(x,t) - V^\varepsilon(x,t))\varepsilon(\zeta_t^\varepsilon)$ tends to zero in probability in $L^2(\mathbb{R}^n \times (0,T))$.

The function $V^\varepsilon$ as an element of $L^2(\mathbb{R}^n \times (0,T))$, depends continuously on the trajectories of the process $\zeta_\cdot^\varepsilon$ in the topology of $C(0,T)$, so does the product $V^\varepsilon(x,t)\exp(\zeta_t^\varepsilon)$. Therefore, convergence in law of the process $\zeta_\cdot^\varepsilon$ towards $\sigma W_\cdot$ implies convergence in law of the expression $V^\varepsilon \exp(\zeta_\cdot^\varepsilon)$ to the function $v^0 \exp(\sigma W_\cdot)$, where $v^0$ is a solution to problem (35).

It remain to show that $v^0 \exp(\sigma W_\cdot$ solves the homogenized equation (36). To this end we denote $\hat{u} = v^0 \exp(\sigma W_\cdot)$ and

$$\hat{A} = \hat{a}_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \hat{b}_i \frac{\partial}{\partial x_i} + \hat{G},$$

and consider for arbitrary $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ the inner product $(\hat{u}(t), \varphi)$ taken in $L^2(\mathbb{R}^n)$. This expression defines a diffusion process in $\mathbb{R}$. Applying Ito's formula to this process gives

$$d(\hat{u}, \varphi) = \exp(\sigma W_t) d(v^0(t), \varphi) + \sigma(v^0(t), \varphi)\exp(\sigma W_t) dW_t$$

$$+\frac{1}{2}\sigma^2 (v^0(t), \varphi)\exp(\sigma W_t) dt = \exp(\sigma W_t)(\hat{A} v^0(t), \varphi) dt$$

$$(f(x,t), \varphi)\exp(\sigma W_t)\exp(-\sigma W_t) dt + \sigma(\hat{u}(t), \varphi) dW_t + \frac{1}{2}\sigma^2(\hat{u}(t), \varphi) dt$$

$$= (\hat{A}\hat{u}(t), \varphi) dt + \frac{1}{2}\sigma^2(\hat{u}(t), \varphi) dt + (f(x,t), \varphi) dt + \sigma(\hat{u}(t), \varphi) dW_t.$$

Considering also an evident relation $\hat{u}(0) = u_0$ we conclude that $\hat{u}$ is a solution of problem (36). According to [3] this problem has a unique solution, thus the limit law is uniquely defined. □

In the end of this section we formulate similar results for initial boundary problems. Given a Lipschitz domain $Q \subset \mathbb{R}^n$, consider in the cylinder $Q \times (0,T)$ a Dirichlet initial boundary problem of the form

$$\frac{\partial}{\partial t} u^\varepsilon(x,t) = \operatorname{div}\left( a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) \nabla u^\varepsilon(x,t) \right) + \frac{1}{\varepsilon} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) u^\varepsilon(x,t) + f(x,t), \quad (37)$$

$$u^\varepsilon(x,t) = 0 \text{ on } \partial Q \times (0,T), \qquad u(x,0) = u_0(x),$$

where $u_0 \in L^2(Q)$ and $f \in L^2(Q \times (0,T))$. Under assumption **H3** for each $\varepsilon > 0$ the existence and the uniqueness of a solution of this problem in the space $L^2((0,T); H_0^1(Q)) \cap C((0,T); L^2(Q))$ are well known, see, for instance, [8].

The statement below can be justified in the same way as that for the case of Cauchy problem. We omit its proof.

**Theorem 5.** *Let conditions* **H1**–**H5** *be fulfilled. Then, as* $\varepsilon \to 0$*, a solution* $u^\varepsilon$ *of problem (37) converges in law in the strong topology of* $L^2(Q \times (0,T))$ *to a solution of the limit (homogenized) stochastic partial differential equation which has the form*

$$d\hat{u} = \Big(\hat{a}_{ij}\frac{\partial^2 \hat{u}}{\partial x_i \partial x_j} + \hat{b}_i \frac{\partial \hat{u}}{\partial x_i} + \hat{g}\hat{u}\Big)dt + \sigma \hat{u}\, dW_t + f(x,t), \tag{38}$$

$$\hat{u}(x,t) = 0 \ \text{on}\ \partial Q \times (0,T), \quad \hat{u}(x,0) = u_0(x).$$

*All the coefficients of this equation are the same as in Theorem 4.*

Similar results hold true for Neumann and Fourier initial boundary problems.

## 7. Equations with diffusion driving process

In this section we consider an important particular case of a diffusion finite dimensional driving process in (1). Then problem (2)–(3) reads

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\Big(a(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})\nabla u^\varepsilon(x,t)\Big) + \frac{1}{\varepsilon}g(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}})u^\varepsilon(x,t) + f(x,t), \tag{39}$$

$$u^\varepsilon(x,0) = u_0(x);$$

here and afterwards $\xi_s$ is a stationary diffusion process with values in $\mathbb{R}^d$. We denote the generator of this process by $\mathcal{L}$:

$$\mathcal{L} = q_{km}(y)\frac{\partial^2}{\partial y_k \partial y_m} + B(y)\cdot \nabla_y.$$

The advantages of operators with diffusion driving processes are
- the coefficients of homogenized problem can be found in terms of solutions of non random elliptic auxiliary problems;
- sufficient conditions for mixing properties required in **H5** can be formulated explicitly in terms of the coefficients of generator $\mathcal{L}$.

We suppose the following conditions to hold

**A1.** The coefficients $a(z,y)$, $g(z,y)$ and $q(y)$ are uniformly bounded as well as their derivatives: there exists $C>0$ such that for all $(z,y) \in \mathbb{T}^n \times \mathbb{R}^d$

$$|a_{ij}(z,y)| + |\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \le C\,,$$
$$|g(z,y)| + |\nabla_z g(z,y)| + |\nabla_y g(z,y)| \le C\,,$$
$$|q_{km}(y)| + |\nabla_y q_{km}(y)| \le C\,,$$

for all $(z,y) \in \mathbb{T}^n \times \mathbb{R}^d$ and for all $1 \le i,j \le n$, $1 \le k,l \le d$; the symbols $\nabla_z$ and $\nabla_y$ stand for the gradients with respect to $z$ and $y$ respectively.
The vector function $B$ and its derivatives satisfy polynomial growth condition:

$$|B(y)| + |\nabla B(y)| \le C(1+|y|)^{\mu}$$

for some $\mu \ge 0$ and $C > 0$.
**A2.** Matrices $a_{ij}$ and $q_{km}$ are uniformly positive definite: there is $\lambda > 0$ such that

$$\lambda\,|z'|^2 \le a_{ij}(z,y)z'_i z'_j\,, \ \forall z' \in \mathbb{R}^n\,,$$
$$\lambda\,|y'|^2 \le q_{km}(z,y)y'_k y'_m\,, \ \forall y' \in \mathbb{R}^d\,.$$

**A3.** There exist constants $\alpha > -1$, $R > 0$ and $C > 0$ such that

$$\frac{b(y)\cdot y}{|y|} \le -C|y|^{\alpha} \qquad \text{for all } y \in \{y\,:\,|y| \ge R\}.$$

**A4.** Centering condition: $\mathbf{E}\int_{\mathbb{T}}^{n} g(z,\xi_s)dz = 0$.

As was proved in [11], under conditions **A1**–**A3** the process $\xi_s$ has a unique invariant measure in $\mathbb{R}^d$ whose density solves the problem

$$\mathcal{L}^*\rho = 0, \qquad \int\limits_{\mathbb{R}^d} \rho(y)dy = 1;$$

the notation $\mathcal{L}^*$ is used for the adjoint operator. Moreover, for any $N > 0$ there is $C_N > 0$ such that

$$\rho(y) \le C_N(1+|y|)^{-N}.$$

It was also shown in the same work that the strong mixing coefficient of a stationary version of the diffusion process $\xi_s$ possesses the property **H4**.

Denote $L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$ the weighted $L^2$ space with the norm

$$\|f(z,y)\|^2_\rho = \int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} f^2(z,y)\rho(y)dydz,$$

and

$$H^1_\rho(\mathbb{T}^n \times \mathbb{R}^d) = \{f \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)\,:\,|\nabla_z f| + |\nabla_y f| \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)\}.$$

Also we will use the notation $\mathcal{A} = \mathrm{div}_z a(z,y)\nabla_z$. The proof of following two technical statements can be found in [2].

**Lemma 4.** *Let $f \in L^2_\rho(\mathbb{T}^n \times \mathbb{R}^d)$, and suppose that*

$$|f(z,y)| \le C\,(1+|y|)^p \ \ \forall (z,y) \in \mathbb{T}^n \times \mathbb{R}^d$$

*for some $C > 0$ and $p \in \mathbb{R}$ and that $\int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} f(z,y)\rho(y)dydz = 0$. Then the equation*

$$(\mathcal{A}+\mathcal{L})\, u(z,y) = f(z,y)$$

*has a solution in the space of functions of polynomial growth in $y$:*

$$|u(z,y)| \le C\,(1+|y|)^{p+1} \;\; \forall (z,y)\in \mathbb{T}^n\times\mathbb{R}^d$$

*This solution is unique up to an additive constant.*
*If, in addition, there is $N>0$ such that for all $n_1$, $n_2\in\mathbb{N}$ with $n_1+n_2\le N$ we have*

$$\left|\partial_z^{n_1}\,\partial_y^{n_2} f(z,y)\right| \le C\,(1+|y|)^p \;\; \forall (z,y)\in \mathbb{T}^n\times\mathbb{R}^d,$$

*then*

$$\left|\partial_z^{n_1}\,\partial_y^{n_2} u(z,y)\right| \le C\,(1+|y|^{p+1}) \;\; \forall (z,y)\in \mathbb{T}^n\times\mathbb{R}^d.$$

**Proposition 5.** *For any $t>0$, $\mu>0$, $\gamma>0$ and $\beta>0$ the relation holds*

$$\lim_{\varepsilon\to 0}\mathbf{E}\Big(\sup_{s\le t}\varepsilon^\beta|\xi_{s/\varepsilon^\gamma}|^\mu\Big)=0.$$

We proceed with averaging procedure. As above we represent $g(z,\xi_s)$ as the sum $g(z,\xi_s) =< g > (\xi_s) + \tilde g(z,\xi_s)$ with $< g > (\xi_s) = \int_{\mathbb{T}^n} g(z,\xi_s)dz$. In order to construct the limit operator we need two correctors which are defined as solutions of the following auxiliary equations

$$(\mathcal{A}+\mathcal{L})\chi_j(z,y) = -\frac{\partial}{\partial z_i}a_{ij}(z,y), \tag{40}$$

$$(\mathcal{A}+\mathcal{L})G(z,y) = -\tilde g(z,y). \tag{41}$$

By Lemma 4 these equations have solutions in $H^1_\rho(\mathbb{T}^n\times\mathbb{R}^d)$ of polynomial growth in $y$.

The main result of this section is

**Theorem 6.** *Under assumptions* **A1**–**A4** *the solution $u^\varepsilon$ of problem (39) converges in law, as $\varepsilon\to 0$, in the space $L^2((0,T)\times\mathbb{R}^n)$ to a solution $u^0$ of the following stochastic PDE*

$$du^0(x,t) = \Big(\mathrm{div}(\hat a\nabla u^0(x,t)) - \hat b\nabla u^0(x,t)\hat g u^0(x,t)\Big)dt + \sigma u^0(x,t)dW_t,$$

$$u^0(x,0)=u_0,$$

*where*

$$\hat a = \int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} a(z,y)(\mathbf{Id}+\nabla_z\chi(z,y))\rho(y)dzdy,$$

$$\hat b = \int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d}(g(z,y)\chi(z,y) + a(z,y)\nabla_z G(z,y))\rho(y)dzdy,$$

$$\hat{g} = \int\limits_{\mathbb{T}^n}\int\limits_{\mathbb{R}^d} g(z,y)G(z,y)\rho(y)dzdy + \frac{1}{2}\sigma^2, \tag{42}$$

$$\sigma^2 = \int\limits_{\mathbb{R}^d} 2q(y)\Big(\int\limits_{\mathbb{T}^n} \nabla_y G(z,y))dz\Big)\cdot\Big(\int\limits_{\mathbb{T}^n} \nabla_y G(z,y))dz\Big)\rho(y)dy,$$

*and $W_t$ is a standard 1D Wiener process.*

*Proof.* Since all the conditions of Theorem 4 are fulfilled, we need not prove the convergence of solutions of problem (39) to a solution of a limit stochastic PDE but only the fact that the effective coefficients given by (42) coincide with those defined in Theorem 4.

First we show that the corresponding diffusion coefficients $\sigma$ are identical. The validity of the Central Limit Theorem for stationary processes of the form $F(\xi_s)$, $F \in L^q(\mathbb{R}^d)$, with diffusion $\xi_s$ satisfying condition **A3**, have been justified in [11]. In particular, it has been proved that for the process $< g(\xi_s) >$ the corresponding variance $\sigma$ is given by the formula

$$\sigma^2 = \int\limits_{\mathbb{R}^d} 2q(y)\nabla\bar{G}(y)\cdot\nabla\bar{G}(y)\rho(y)dy,$$

where $\bar{G}$ is a solution to the equation $\mathcal{L}\bar{G}(y) =< g > (y)$. Since operator $\mathcal{L}$ applied to a function $F(z,y)$ commutes with taking the average of $F$ in $z$, we obtain $\nabla < g > (y) = \nabla\int_{\mathbb{T}^n} G(z,y)dz$, and the desired coincidence follows.

We proceed with the other coefficients. For any $\varphi \in C^\infty(0,T;C_0^\infty(\mathbb{R}^n))$ consider the auxiliary process

$$H^\varepsilon(t) = (\varphi, v^\varepsilon) + \varepsilon(\chi^\varepsilon\cdot\nabla\varphi, v^\varepsilon) + \varepsilon(G^\varepsilon, \varphi);$$

here and afterwards for a generic function $F = F(z,y)$ we use the notation $F^\varepsilon(x,t) = F(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2})$; $(\cdot,\cdot)$ stands for the inner product in $L^2(\mathbb{R}^n)$. Applying Ito's formula to $H^\varepsilon(t)$ gives

$$\begin{aligned}
dH^\varepsilon = {} & \{(a^\varepsilon_{ij}\partial_{x_i}\partial_{x_j}\varphi, v^\varepsilon) + \varepsilon^{-1}(\partial_{z_i}a^\varepsilon_{ij}\partial_{x_i}\varphi, v^\varepsilon)\}dt \\
& +\varepsilon^{-1}(\tilde{g}^\varepsilon\varphi, v^\varepsilon)dt + (\nabla_y G^\varepsilon\varphi + \nabla_y\chi^\varepsilon_i\partial_{x_i}\varphi, v^\varepsilon)\cdot\sqrt{q^\varepsilon}dB_t \\
& +\Big\{\varepsilon^{-1}(\mathcal{L}\chi^\varepsilon\nabla\varphi, v^\varepsilon) + (g^\varepsilon\chi^\varepsilon\nabla\varphi, v^\varepsilon) + (\partial_t\varphi, v^\varepsilon) \\
& +\varepsilon^{-1}(\mathcal{A}\chi^\varepsilon\nabla\varphi, v^\varepsilon) + \varepsilon(\chi^\varepsilon\partial_t\nabla\varphi, v^\varepsilon) \\
& +(\partial_{z_i}(a^\varepsilon_{ij}\chi_k)^\varepsilon\partial_{x_j}\partial_{x_k}\varphi + a^\varepsilon_{ij}\partial_{z_i}\chi^\varepsilon_k\partial_{x_j}\partial_{x_k}\varphi, v^\varepsilon) \\
& +\varepsilon(a^\varepsilon\chi^\varepsilon\nabla\nabla\nabla\varphi, v^\varepsilon) + \varepsilon^{-1}(\mathcal{L}G^\varepsilon\varphi, v^\varepsilon) + (G^\varepsilon g^\varepsilon\varphi, v^\varepsilon) \\
& +\varepsilon^{-1}(\mathcal{A}G^\varepsilon\varphi, v^\varepsilon) + (\partial_{z_i}(a^\varepsilon_{ij}G^\varepsilon)\partial_{x_j}\varphi, v^\varepsilon) + (a^\varepsilon_{ij}\partial_{z_i}G^\varepsilon\partial_{x_j}\varphi, v^\varepsilon) \\
& +\varepsilon(a^\varepsilon G^\varepsilon\nabla\nabla\varphi, v^\varepsilon)\Big\}dt.
\end{aligned}$$

Taking into account the equations (40), (41) we get

$$
\begin{aligned}
dH^\varepsilon = \Big\{ & \{(a^\varepsilon_{ij}\partial_{x_i}\partial_{x_j}\varphi, v^\varepsilon) + (\partial_{z_i}(a^\varepsilon_{ij}\chi_k)^\varepsilon\partial_{x_j}\partial_{x_k}\varphi + a^\varepsilon_{ij}\partial_{z_i}\chi^\varepsilon_k\partial_{x_j}\partial_{x_k}\varphi, v^\varepsilon) \\
& +(g^\varepsilon\chi^\varepsilon\nabla\varphi, v^\varepsilon) + (G^\varepsilon g^\varepsilon\varphi, v^\varepsilon) + (\partial_t\varphi, v^\varepsilon) \\
& +(\partial_{z_i}(a^\varepsilon_{ij}G^\varepsilon)\partial_{x_j}\varphi, v^\varepsilon) + (a^\varepsilon_{ij}\partial_{z_i}G^\varepsilon\partial_{x_j}\varphi, v^\varepsilon)\Big\}dt \qquad (43)\\
& (\nabla_y G^\varepsilon\varphi + \nabla_y\chi^\varepsilon_i\partial_{x_i}\varphi, v^\varepsilon)\cdot\sqrt{q^\varepsilon}dB_t + \varepsilon dR^\varepsilon(t),
\end{aligned}
$$

where $B_t$ is a standard $n$-dimensional Brownian motion, and $R^\varepsilon$ satisfies the estimate

$$\mathbf{E}\sup_{t\le T}|R^\varepsilon(t)| \le C.$$

This estimate follows from Proposition 1.

For the stochastic term on the right hand side of (43) we have

**Lemma 5.** *The bound holds*

$$\lim_{\varepsilon\to 0}\mathbf{E}\sup_{t\le T}\Big|\int_0^t \Big(\nabla_y G^\varepsilon(\cdot,s)\varphi(\cdot,s)$$

$$+\nabla_y\chi_i\varepsilon(\cdot,s)\partial_{x_i}\varphi(\cdot,s), v^\varepsilon(\cdot,s)\Big)\cdot\sqrt{q^\varepsilon(s)}dB_s\Big| = 0.$$

*Proof.* Since $\int_{\mathbb{T}^n}\nabla_y\chi(z,y)dz = 0$ and $\int_{\mathbb{T}^n}\nabla_y G(z,y)dz = 0$, we have

$$\mathbf{E}\sup_{t\le T}\Big|\int_0^t \Big(\nabla_y G^\varepsilon(\cdot,s)\varphi(\cdot,s) + \nabla_y\chi^\varepsilon_i(\cdot,s)\partial_{x_i}\varphi(\cdot,s), v^\varepsilon(\cdot,s)\Big)\cdot\sqrt{q^\varepsilon(s)}dB_s\Big|$$

$$= \varepsilon\mathbf{E}\sup_{t\le T}\Big|\int_0^t \Big(J^{1,\varepsilon}_i(\cdot,s), \partial_{x_i}(\varphi(\cdot,s)v^\varepsilon(\cdot,s))$$

$$+J^{2,\varepsilon}_{ij}(\cdot,s), \partial_{x_j}(\partial_{x_i}\varphi(\cdot,s)v^\varepsilon(\cdot,s))\Big)\cdot\sqrt{q^\varepsilon(s)}dB_s\Big|$$

where $J^1(z,y)$ and $J^2(z,y)$ are periodic in $z$ functions such that

$$\mathrm{div}_z J^1(z,y) = \nabla_y G(z,y), \quad \mathrm{div}_z J^2(z,y) = \nabla_y\chi(z,y).$$

The statement of the lemma now follows from (1) and the Burkholder-Davis-Gundy inequality. □

To complete the proof of the theorem we notice that by virtue of (1) and the Birkhoff theorem

$$\lim_{\varepsilon\to 0}\mathbf{E}\sup_{t\le T}\Big|(\varphi(\cdot,t), v^\varepsilon(\cdot,t)) - (\varphi(\cdot,0), u_0) - \int_0^t(\partial_s\varphi(\cdot,s), v^\varepsilon(\cdot,s))ds$$

$$-\int_0^t\Big(\Big\{\hat a_{ij}\partial_{x_i}\partial_{x_j}\varphi(\cdot,s) + \hat b\nabla\varphi(\cdot,s) + \bar G\varphi(\cdot,s)\Big\}, v^\varepsilon(\cdot,s)\Big)ds\Big| = 0,$$

where the coefficients $\hat{a}$, $\hat{b}$ have been defined in Theorem 6 and

$$\hat{G} = \int_{\mathbb{T}^n}\int_{\mathbb{R}^d} g(z,y)G(z,y)\rho(y)dzdy. \qquad \square$$

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# Part II

## Problems with concentration

# $\Gamma$-convergence for concentration problems

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## 1 Introduction

In these notes we will consider a large class of variational problems in which a concentration phenomenon occurs. This is a typical phenomenon in problems with scaling invariance as, for instance, semilinear problems involving the critical Sobolev exponent. More precisely we will study the asymptotic behaviour of problems of the form

$$S_\varepsilon^F(\Omega) := \sup \left\{ \varepsilon^{2^*} \int_\Omega F(u)\, dx \ : \ \int_\Omega |\nabla u|^2 dx \le \varepsilon^2 \,, \ u = 0 \text{ on } \partial\Omega \right\}, \quad (1)$$

when $\varepsilon \to 0^+$, where $\Omega$ is a bounded domain of $\mathbf{R}^n$, $n \ge 3$, and $F$ is a nonnegative upper-semicontinuous function bounded from above by $c|t|^{2^*}$, with $2^* = \frac{2n}{n-2}$ being the critical Sobolev exponent.

If $F$ is a smooth function the maximizers of (1) satisfy the following semilinear equation

$$\begin{cases} -\Delta u = \lambda_\varepsilon f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f = F'$ and the Lagrange multiplier $\lambda_\varepsilon$ tends to $+\infty$ when $\varepsilon \to 0^+$.

The, by now, classical approach for this type of semilinear problems is the *concentration-compactness alternative* due to P.L. Lions [21]. For the case of smooth $F$ with critical growth (e.g. $F(t) = |t|^{2^*}$) he proved that the sequence of maximizers either concentrates at a single point (in a sense that will be clear in Section 2) or is compact in $H_0^1(\Omega)$. In particular in the case $\Omega \ne \mathbf{R}^n$ one can exclude compactness and deduce that only concentration is allowed.

As a particular case of problem (1) with, possibly, non smooth (or degenerate) $F$, we can also obtain interesting free boundary problems as the *plasma problem* or the *Bernoulli free boundary problem* (see for instance [16], [14] and [9]). The latter can be consider as the weak Euler-Lagrange equation of the following variational problem

$$S_\varepsilon^V = \sup\left\{\varepsilon^{-2^*}|A| \,:\, A \subseteq \Omega\,,\ \mathrm{cap}(A,\Omega) \le \varepsilon^2\right\}\,,$$

where $\mathrm{cap}(A,\Omega)$ denotes the harmonic capacity of the set $A$. The critical case, corresponding to the choice $F(t) = |t|^{2^*}$, and the Bernoulli problem will be described in details in Section 2 and can be considered as two "extreme" particular cases for problem (1).

In order to deal with the general problem (1), the concentration-compactness alternative has been generalized by Flucher and Müller in [13]. Again one deduce concentration for the sequence of maximizers (or almost maximizers).

In this notes we propose a different method to deduce concentration and, more in general, to study the asymptotic behaviour of problem (1).

We will use the notion of $\Gamma$-convergence which is the natural convergence for functionals in order to deduce convergence of extrema.

The idea of $\Gamma$-convergence is to substitute a sequence of functionals $\{\mathcal{F}_\varepsilon\}$ by an effective *$\Gamma$-limit* functional $\mathcal{F}$ which captures the relevant features of the sequence $\{\mathcal{F}_\varepsilon\}$. In particular, sequences of (almost) maximizers converge to maximum points of $\mathcal{F}$. Therefore, in terms of $\Gamma$-convergence, we study the asymptotic behaviour of the functional

$$\mathcal{F}_\varepsilon(u) = \varepsilon^{-2^*}\int_\Omega F(\varepsilon u)\,dx\,,$$

with the constraint $\int_\Omega |\nabla u|^2 dx \le 1$ and $u \in H_0^1(\Omega)$.

Analyzing the $\Gamma$-limit $\mathcal{F}$ we describe the asymptotic behaviour of $\mathcal{F}_\varepsilon(u_\varepsilon)$ along all weakly converging sequences and we also deduce concentration.

The approach of $\Gamma$-convergence for this kind of concentration phenomena is recent and has been successfully used also in the study of Ginzburg-Landau problem by Alberti, Baldo and Orlandi [1]. A delicate point, in general for $\Gamma$-convergence and here in particular, is the choice of the topology which should be strong enough in order to assure convergence of maxima and weak enough in order to detect concentration.

A second natural question for this type of concentration results is whether it is possible to characterize the concentration point, in particular whether its position can be influenced by the shape of the domain.

A crucial role in the identification of the concentration point is played by the Green's function for the Dirichlet problem with the Laplacian. More precisely we will see that the maximizing sequences concentrates at the *harmonic centers* of $\Omega$ (the minima of the *Robin function*; i.e., the diagonal of the regular part of the Green's function).

## 2 Two classical examples

We introduce the problem starting with two examples: the Sobolev inequality and the harmonic capacity.

## 2.1 Sobolev inequality

Let us fix a domain $\Omega \subset \mathbf{R}^n$, with $n \geq 3$. We know that the Sobolev space $H_0^1(\Omega)$ is embedded continuously in $L^p(\Omega)$ for every $p \leq 2^* = \frac{2n}{n-2}$; i.e., there exists a constant $C_p(\Omega)$, depending on $p$ and $\Omega$, such that

$$\int_\Omega |u|^p dx \leq C_p(\Omega) \left( \int_\Omega |\nabla u|^2 dx \right)^{\frac{p}{2}} \tag{2}$$

for every $u \in H_0^1(\Omega)$. The embedding is also compact for $p < 2^*$. It is well known that for $p = 2^*$ (the so-called *critical case*) the embedding is not compact.

There is a large number of interesting and difficult analytical and geometrical problems involving the critical growth and many interesting phenomena arise from this lack of compactness.

Let us consider first the Sobolev inequality in $\mathbf{R}^n$

$$\int_{\mathbf{R}^n} |u|^{2^*} dx \leq S^* \left( \int_{\mathbf{R}^n} |\nabla u|^2 dx \right)^{\frac{2^*}{2}}, \qquad \forall u \in C_0^\infty(\mathbf{R}^n) . \tag{3}$$

Inequality (3) holds true for all functions in the Deny space $D^{1,2}(\mathbf{R}^n)$ obtained as the closure of $C_0^\infty(\mathbf{R}^n)$ with respect to the topology induced by the $L^2$ norm of the gradient ($D^{1,2}(\mathbf{R}^n) = \overline{C_0^\infty(\mathbf{R}^n)}^{\|\nabla u\|_{L^2}}$) and $S^*$ is the best Sobolev constant.

**Question 1** Is $S^*$ achieved? In other words, does there exist a function $u \in D^{1,2}(\mathbf{R}^n)$ such that (3) holds with equality?

In the case of $\mathbf{R}^n$ the answer is *yes*. This problem has been solved by Talenti in '76 ([23]), and all the possible solutions have been completely characterized. Namely, all the solutions of the following variational problem

$$S^* = \max \left\{ \frac{\displaystyle\int_{\mathbf{R}^n} |u|^{2^*} dx}{\left( \displaystyle\int_{\mathbf{R}^n} |\nabla u|^2 \right)^{\frac{2^*}{2}}} \ : \ u \in D^{1,2}(\mathbf{R}^n) \right\},$$

or equivalently

$$S^* = \max \left\{ \int_{\mathbf{R}^n} |u|^{2^*} dx \ : \ u \in D^{1,2}(\mathbf{R}^n) \text{ and } \int_{\mathbf{R}^n} |\nabla u|^2 \leq 1 \right\}, \tag{4}$$

have been characterized.

Indeed, one can consider the Euler-Lagrange equation of (4) and, by a rearrangement argument, prove that

$$u_1(x) = \frac{1}{(c^2 + |x|^2)^{\frac{n-2}{2}}}\,, \tag{5}$$

is a solution. Here the constant $c$ is a renormalization which gives

$$\int_{\mathbf{R}^n} |\nabla u_1|^2 dx = 1\,.$$

All other solutions can be obtained by scaling and translating $u_1$. Indeed, both the Dirichlet integral on $\mathbf{R}^n$ and the $L^{2^*}$ norm are invariant under the following dilations and translations:

$$u(x) = \sigma^{-\frac{n}{2^*}} u_1\left(\frac{x-y}{\sigma}\right) \qquad \text{for } \ \sigma > 0 \ \text{ and } \ y \in \mathbf{R}^n\,.$$

Namely $|\nabla u(x)|^2 = \sigma^{-n}|\nabla u_1\left(\frac{x-y}{\sigma}\right)|^2$ and $|u(x)|^{2^*} = \sigma^{-n}|u_1\left(\frac{x-y}{\sigma}\right)|^{2^*}$, and thus, by a change of variables,

$$\int_{\mathbf{R}^n} |\nabla u|^2 dx = \int_{\mathbf{R}^n} |\nabla u_1|^2 dx \qquad \text{and} \qquad \int_{\mathbf{R}^n} |u|^{2^*} dx = \int_{\mathbf{R}^n} |u_1|^{2^*} dx\,.$$

Let us consider now the case $\Omega \neq \mathbf{R}^n$, for instance consider the case $\Omega$ bounded. In this case $H_0^1(\Omega) = D^{1,2}(\Omega)$ and the Sobolev inequality reads

$$\int_{\Omega} |u|^{2^*} dx \le S^*(\Omega)\left(\int_{\Omega} |\nabla u|^2 dx\right)^{\frac{2^*}{2}} \qquad \forall u \in H_0^1(\Omega)\,. \tag{6}$$

*Remark 1.* Again by a scaling argument one can see that the best Sobolev constant does not depend on the domain; i.e.,

$$S^*(\Omega) = S^* \qquad \forall\ \Omega \subseteq \mathbf{R}^n\,.$$

**Question 2** Is $S^*$ achieved in $\Omega$?

The answer in this case is *no*. In fact, otherwise, we would find solutions for (4) with compact support and hence not of the form (5). Thus an other question arise naturally.

**Question 3** What we can expect from an *optimal sequence*; i.e., a sequence $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\frac{\int_{\Omega} |u_\varepsilon|^{2^*} dx}{\left(\int_{\Omega} |\nabla u_\varepsilon|^2 dx\right)^{\frac{2^*}{2}}} = S^* + o(1)\,? \tag{7}$$

We may have a rather precise idea of the situation through the following example: take the sequence $v_\varepsilon$ such that $v_\varepsilon(|x|) = \varepsilon^{-\frac{n}{2^*}} u_1(\frac{x}{\varepsilon})$ and $\Omega = B_R$ ($B_R$ denotes the ball centered in the origin of radius $R$). Clearly we have

$$\lim_{\varepsilon\to 0}\int_{B_R} |v_\varepsilon(|x|)|^{2^*}\,dx = \lim_{\varepsilon\to 0}\int_{B_{\varepsilon R}} |u_1|^{2^*}\,dx = S^*\,.$$

In particular we get that the sequence $|v_\varepsilon|^{2^*}$ converges to $S^*\delta_0$ ($\delta_0$ denotes the Dirac mass at zero) weakly* in the sense of measure. It is then easy to see that $u_\varepsilon(x) := (v_\varepsilon(|x|) - v_\varepsilon(R))_+ \in H^1_0(\Omega)$ satisfies (7). Moreover also $u_\varepsilon$ concentrates all the energy at zero; i.e., $|u_\varepsilon|^{2^*} \overset{*}{\rightharpoonup} S^*\delta_0$ and $|\nabla u_\varepsilon|^2 \overset{*}{\rightharpoonup} \delta_0$.

The scaling invariance of the problem is responsible for this phenomenon of concentration and in general for the lack of compactness in the embedding theorem. This lack of compactness can be very well described, and this has been done by P.L. Lions ([21]) by means of the concentration-compactness principle. Generally speaking, it consists in the analysis of the possible ways a bounded sequence of measures can loose compactness. In the special case of the Sobolev embedding theorem he proves in particular the following *Concentration-compactness alternative.*

We fix a sequence $u_\varepsilon \in D^{1,2}(\Omega)$, with $\|\nabla u_\varepsilon\|_{L^2} \le 1$. Up to a subsequence there exists a function $u_0 \in D^{1,2}(\Omega)$ such that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^{2^*}(\Omega) \qquad \text{and} \qquad \nabla u_\varepsilon \rightharpoonup \nabla u_0 \quad \text{in} \quad L^2(\Omega)\,;$$

i.e., $u_\varepsilon \rightharpoonup u_0$ in $D^{1,2}(\Omega)$. We may also assume that there exist two measures $\mu^*, \nu^* \in \mathcal{M}(\overline{\Omega}) := (C(\overline{\Omega}))'$, such that

$$|u_\varepsilon|^{2^*}dx \overset{*}{\rightharpoonup} \nu^* \quad \text{in} \quad \mathcal{M}(\overline{\Omega}) \qquad \text{and} \qquad |\nabla u_\varepsilon|^2 dx \overset{*}{\rightharpoonup} \mu^* \quad \text{in} \quad \mathcal{M}(\overline{\Omega})\,.$$ [1]

In order to study the possible lack of compactness for $u_\varepsilon$ the idea of P.L. Lions is to characterize the measures $\nu^*$ in terms of $\mu^*$.

*Remark 2.* Note that if $\nu^* = |u_0|^{2^*}dx$ then we have compactness in $L^{2^*}$ for $u_\varepsilon$, while if we also know that $\mu^* = |\nabla u_0|^2 dx$ we conclude strong convergence of $u_\varepsilon$ in $D^{1,2}(\Omega)$.

In general, by the lower semi-continuity of the norm, we get

$$\mu^* \ge |\nabla u_0|^2\,dx\,.$$

Thus we can isolate the atoms of $\mu^*$, $\{x_i\}_{i\in J}$, and rewrite $\mu^*$ as follows

[1] Recall: we say that a sequence of measures $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$ if and only if

$$\lim_{\varepsilon\to 0}\int_{\overline{\Omega}} \varphi\,d\mu_\varepsilon = \int_{\overline{\Omega}} \varphi\,d\mu \qquad \forall\,\varphi \in C(\overline{\Omega})\,.$$

$$\mu^* = |\nabla u_0|^2 dx + \sum_{i\in J} \mu_i \delta_{x_i} + \widetilde{\mu}\,, \tag{8}$$

where $\mu_i$ denotes the positive weight of the atom $x_i$ and $\widetilde{\mu}$ is the non-atomic part of $\mu^* - |\nabla u_0|^2 dx$.

*Remark 3.* Note that $\widetilde{\mu}$ in general may also contain a part which is absolutely continuous with respect to the Lebesgue measure.

**Lemma 1** ([21]) *Let $u_\varepsilon, u_0 \in D^{1,2}(\Omega)$ be such that $\int_\Omega |\nabla u_\varepsilon|^2 \le 1$, $u_\varepsilon \rightharpoonup u_0$ in $D^{1,2}(\Omega)$, $|u_\varepsilon|^{2^*} dx \overset{*}{\rightharpoonup} \nu^*$ and $|\nabla u_\varepsilon|^2 dx \overset{*}{\rightharpoonup} \mu^*$ in $\mathcal{M}(\overline{\Omega})$ for some measures $\mu^*$ and $\nu^*$. Assume $\mu^*$ be decomposed as in (8); then*

1. *there exist non-negative constants $\nu_i^*$ such that*
   *i)* $\nu^* = |u_0|^{2^*} dx + \sum_{i\in J} \nu_i^* \delta_{x_i}$;
   *ii)* $|u_\varepsilon - u_0|^{2^*} dx \overset{*}{\rightharpoonup} \sum_{i\in J} \nu_i^* \delta_{x_i}$ *in* $\mathcal{M}(\overline{\Omega})$;
   *iii)* $0 \le \nu_i^* \le S^* (\mu_i)^{\frac{n}{n-2}}$ *for all* $i \in J$.
2. *(Alternative) If $\nu^*(\overline{\Omega}) = S^*$ and $\mu^*(\overline{\Omega}) = 1$; i.e., $u_\varepsilon$ is an optimal sequence for the Sobolev embedding, then one of the two following situations is possible*
   *a) Concentration: there exists $x_0 \in \overline{\Omega}$ such that $\mu^* = \delta_{x_0}$ and $\nu^* = S^* \delta_{x_0}$;*
   *b) Compactness: $\nu^* = |u_0|^{2^*} dx$.*

Note that in the alternative b)), by the optimality of $u_\varepsilon$ we also get $\mu^* = |\nabla u_0|^2 dx$ and hence $u_\varepsilon \to u_0$ in $D^{1,2}(\Omega)$.

*Remark 4.*

1. In the case $\Omega \neq \mathbf{R}^n$, by the fact that the Sobolev constant is never achieved in $\Omega$, we deduce that only alternative a) is possible; i.e., concentration occurs.
2. Part 1 of Lemma 1 is obtained by a fine use of the Sobolev inequality. In particular note that ii) states that the only way a bounded sequence in $D^{1,2}(\Omega)$ can loose compactness in $L^{2^*}(\Omega)$ is by concentration (so an oscillating bounded sequence for which the gradients weak converge but do not concentrate, is always strongly convergent in $L^{2^*}(\Omega)$).

### 2.2 Capacity and isoperimetric inequality for the capacity

We now introduce the notion of *harmonic capacity* for which we will see that a similar concentration phenomenon arises.

**Definition 2** *(Capacity) Given $\Omega \subseteq \mathbf{R}^n$ and an open set $A \subset \Omega$, the capacity of $A$ with respect to $\Omega$ is the following set function*

$$\operatorname{cap}(A, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^2 dx \,:\, u \geq 1 \text{ a.e. in } A\,,\ u \in H_0^1(\Omega) \right\}. \tag{9}$$

Note that the constraint $u \geq 1$ a.e. in $A$ in the definition of the capacity is convex and strongly closed in $H_0^1(\Omega)$ and hence it is closed in the weak topology. So that there exists a unique minimum point for problem (9). It is called the *capacitary potential of $A$ with respect to $\Omega$*.

*Remark 5.*

1. By a truncation argument we can show that the capacitary potential $u_A$ satisfies $u_A = 1$ a.e. in $A$.
2. The capacitary potential is also a weak solution of the following Euler-Lagrange equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{A} \\ u = 1 & \text{on } \partial A \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{10}$$

The capacity theory is a classical tool in potential theory. In the variational approach it is the natural object when studying problems in the Sobolev space $H_0^1(\Omega)$. It is involved in regularity results for elliptic problems, fine behaviour of Sobolev functions, etc. (for a very nice review of this subject see Frehse [17]).

In general it is not very easy to compute explicitly the capacity of a set. Let us consider the easiest case: two concentric balls.

*Example 1.* Fix $0 < r < R$ and compute the capacity of $B_r$ with respect to $B_R$. From now on we will denote by

$$K(\rho) = \frac{\gamma_n}{\rho^{n-2}} \qquad \text{with } \gamma_n = \frac{1}{(n-2)|S^{n-1}|}$$

the *fundamental singularity of the Laplacian $\Delta$ in $\mathbf{R}^n$*, $n \geq 3$, and then $K(|x-y|)$ will be the *fundamental solution with singularity at $y$*. In particular $K(|x|)$ is harmonic outside zero. Moreover $K(|x|) - K(R) = 0$ on $\partial B_R$ and $\frac{K(|x|)-K(R)}{K(r)-K(R)} = 1$ on $\partial B_r$. Thus we have that the capacitary potential of $B_r$ with respect to $B_R$ is given by

$$u(x) = \min\left\{ \frac{K(|x|) - K(R)}{K(r) - K(R)}, 1 \right\}.$$

Then, using the fact that $-\Delta K(|x|) = \delta_0$ in the sense of distributions, we get

$$\mathrm{cap}(B_r, B_R) = \int_{B_R} |\nabla u|^2 dx = \frac{1}{K(r) - K(R)} \int_{B_R \setminus B_r} \nabla K \nabla u \, dx$$
$$= \frac{1}{K(r) - K(R)} .$$

Similarly, we obtain

$$\mathrm{cap}(B_r, \mathbf{R}^n) = \frac{1}{K(r)} .$$

The notion of capacity can be extended to any subset $E$ of $\Omega$ as follows

$$\mathrm{cap}(E, \Omega) = \inf\{\mathrm{cap}(A, \Omega) \ : \ A \ \text{open}, \quad A \supseteq E\} .$$

**Proposition 3** *Let $A$ and $B$ be two given subsets of $\Omega$. The following properties hold.*

1. *(Monotonicity) If $A \subset B$ then*
$$\mathrm{cap}(A, \Omega) \le \mathrm{cap}(B, \Omega) ;$$
2. *(Subadditivity)*
$$\mathrm{cap}(A \cup B, \Omega) \le \mathrm{cap}(A, \Omega) + \mathrm{cap}(B, \Omega) ;$$
3. *(Scaling property) For any $\rho > 0$ we have*
$$\mathrm{cap}(\rho A, \rho\Omega) = \rho^{n-2} \mathrm{cap}(A, \Omega) ;$$
4. *If $B$ is open and $A \subseteq B \subseteq \Omega$, then*
$$\frac{1}{\mathrm{cap}(A, \Omega)} \ge \frac{1}{\mathrm{cap}(A, B)} + \frac{1}{\mathrm{cap}(B, \Omega)}$$
*and equality holds if and only if $B$ is a superlevel of the capacitary potential $u_A$ of $A$ in $\Omega$.*

The above properties can be easily deduced from the definition of the capacity and making use of the capacitary potentials.

Note that in particular the capacity is an external measure, but in general it is not additive (it can be proved that it is a measure only on the class of zero capacity sets).

The "equivalent" of the Sobolev inequality in the case of the capacity is the so-called *isoperimetric inequality for the capacity*: there exists a constant $S^V(\Omega)$ such that

$$|A| \le S^V(\Omega) \left(\mathrm{cap}(A, \Omega)\right)^{\frac{2^*}{2}} \qquad \forall \ \ A \subseteq \Omega . \tag{11}$$

**Question 4** Is this constant $S^V(\Omega)$ achieved by a non-trivial set $A$? In other words: there exists a subset $A$ of $\Omega$ with $\mathrm{cap}(A, \Omega) > 0$, such that equality holds in (11)?

If $\Omega = \mathbf{R}^n$ the answer is again *yes*. In fact we can also compute it explicitly. It is given by

$$S^V := S^V(\mathbf{R}^n) = \max\{|A| : \quad \mathrm{cap}(A, \mathbf{R}^n) \leq 1\} \ .$$

It easy to see, by a symmetrization argument, that the maximum is achieved on balls; i.e. $A = B_R$. Imposing the constraint $\mathrm{cap}(B_R, \mathbf{R}^n) = 1$ the radius $R$ can be computed explicitly and hence the constant $S^V$ turns out to be the following

$$S^V = \omega_n \gamma_n^{\frac{n}{n-2}} \ , \tag{12}$$

where $\gamma_n = K(1)$ and $\omega_n = |B_1|$.

*Remark 6.* Note that, thanks to the scaling property of the capacity (see Proposition 3, 4)), the equality in the isoperimetric inequality (11) is achieved for all balls. Indeed, if $R$ is such that $\mathrm{cap}(B_R, \mathbf{R}^n) = 1$ and $|B_R| = S^V$, then

$$|B_{\rho R}| = \rho^n |B_R| = S^V \rho^n = S^V \left(\mathrm{cap}(\rho B_R, \mathbf{R}^n)\right)^{\frac{n}{n-2}} \ .$$

In the case $\Omega \neq \mathbf{R}^n$ the answer is *no*. The constant $S^V$ is not achieved by a non trivial set $A$ for a reason which is similar to the one we saw in the Sobolev embedding case. Indeed also in this case it can be seen, using the scaling property of the capacity, that

$$S^V(\Omega) = S^V \qquad \forall\ \Omega \, .$$

Moreover whenever $\mathrm{cap}(\mathbf{R}^n \setminus \Omega, \mathbf{R}^n) \neq 0$ and $\mathrm{cap}(A, \Omega) \neq 0$, it can be seen, by using the maximum principle, that

$$\mathrm{cap}(A, \Omega) > \mathrm{cap}(A, \mathbf{R}^n)$$

and then, for any such $A$, the equality in (11) is not possible.

Thus also in this case we may wonder which is the behaviour of an optimal sequence; i.e., a sequence of sets $A_\varepsilon$ such that

$$\frac{|A_\varepsilon|}{\left(\mathrm{cap}(A_\varepsilon, \Omega)\right)^{\frac{n}{n-2}}} = S^V + o(1) \, . \tag{13}$$

So, as above, to have an idea let us construct an optimal sequence by choosing $A_\varepsilon = B_{r_\varepsilon}$, with $r_\varepsilon \to 0$. Then, for any $R > 0$,

$$\left(\mathrm{cap}(A_\varepsilon, \Omega)\right)^{\frac{n}{n-2}} = \left(\mathrm{cap}(B_{r_\varepsilon}, \Omega)\right)^{\frac{n}{n-2}} = \left(\frac{r_\varepsilon}{R}\right)^n \left(\mathrm{cap}(B_R, \frac{R}{r_\varepsilon}\Omega)\right)^{\frac{n}{n-2}} \ .$$

Now it can be proved that

$$\lim_{\varepsilon \to 0} \mathrm{cap}(B_R, \frac{R}{r_\varepsilon}\Omega) = \mathrm{cap}(B_R, \mathbf{R}^n) \, ,$$

thus, if we choose $R$ such that $|B_R| = S^V$, we clearly have that the sequence satisfies (13).

Notice that the balls $B_{r_\varepsilon}$ shrink to the origin. Again we have a concentration phenomenon for the capacitary potentials similar to the case of the Sobolev embedding.

More in general a way to construct an optimal sequence for (11) is to consider the following variational problem

$$S^V_\varepsilon(\Omega) = \varepsilon^{-2^*} \max\left\{|A| \ : \ \ \mathrm{cap}(A,\Omega) = \varepsilon^2\right\}\,, \tag{14}$$

where the factor $\varepsilon^{-2^*}$ is the right scaling passing from the capacity to the volume.

## 3 The general problem

We are now in a position to introduce a very general class of problems which includes and unifies the two examples shown above as two extreme case of the same phenomenon.

### 3.1 Variational formulation

We will consider the following family of variational problem depending on a small parameter $\varepsilon > 0$

$$S^F_\varepsilon(\Omega) = \varepsilon^{-2^*} \sup\left\{\int_\Omega F(u)\,dx \ : \ u \in D^{1,2}(\Omega)\,,\ \int_\Omega |\nabla u|^2 dx \le \varepsilon^2\right\}, \tag{15}$$

with the following assumptions:

i) $0 \le F(t) \le c|t|^{2^*}$ for every $t \in \mathbf{R}$;
ii) $F \not\equiv 0$ and upper semi-continuous.

To simplify the exposition we also assume

iii) there exist the following two limits

$$F_0(t) = \lim_{t\to 0} \frac{F(t)}{|t|^{2^*}} \quad \text{and} \quad F_\infty(t) = \lim_{t\to\infty} \frac{F(t)}{|t|^{2^*}}\,.$$

With this formulation we recover the two examples seen before.

*Examples.*

1. If $F(t) = |t|^{2^*}$, then (15) gives the Sobolev embedding problem and $S_\varepsilon^F(\Omega) = S^*$ for every $\Omega \subseteq \mathbf{R}^n$.
2. We are allowed to choose $F$ discontinuous, then we recover the case of the capacity taking $F$ of the form
$$F(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1. \end{cases}$$
We claim that, with this choice of $F$, problem (15) coincides with (14); i.e.,
$$S_\varepsilon^V(\Omega) = \sup\left\{ |\{u \geq 1\}| : \quad u \in D^{1,2}(\Omega), \quad \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2 \right\} \quad (16)$$
Clearly
$$S_\varepsilon^V(\Omega) \geq \sup\left\{ |\{u \geq 1\}| : \quad u \in D^{1,2}(\Omega), \quad \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2 \right\};$$
indeed any $u \in D^{1,2}(\Omega)$ satisfying $\int_\Omega |\nabla u|^2 dx \leq \varepsilon^2$ is a good competitor for the computation of the capacity of the set $A = \{u \geq 1\}$ and gives $\mathrm{cap}(A,\Omega) \leq \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2$. On the other hand, if $A$ is an open set such that $\mathrm{cap}(A,\Omega) \leq \varepsilon^2$ and $u_A$ is the capacitary potential of $A$, then $A \subseteq \{u_A \geq 1\}$ and $\int_\Omega |\nabla u_A|^2 dx = \mathrm{cap}(A,\Omega) \leq \varepsilon^2$; hence we get (16).
3. If $F \in C^1$ we can compute the Euler-Lagrange equation and we have
$$\begin{cases} -\Delta u = \lambda_\varepsilon f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
where $f = F'$ and $\lambda_\varepsilon$ is the Lagrange multiplier (one can show that $\lambda_\varepsilon \approx \varepsilon^{-\frac{4}{n-2}}$).
4. In general the non-differentiability of $F$ allows for free boundary problems. The easiest example is again the case of the capacity, whose Euler-Lagrange equation is the so-called *Bernoulli free boundary problem*: look for an open set $A$ and a function $u$ which is the weak solution of
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus A \\ u = 1 & \text{on } \partial A \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = q_\varepsilon & \text{on } \partial A, \end{cases}$$
where $q_\varepsilon$ goes to infinity as $\varepsilon \to 0$ and play the role of the Lagrange multiplier. Another classical example covered by the problem is the so-called *plasma problem* (see e.g. [16]).

### 3.2 Generalized Sobolev inequality

Let us see first some general facts related to the scaling properties of our problem. Denote by $S^F := S_1^F(\mathbf{R}^n)$; i.e.,

$$S^F = \sup\left\{\int_{\mathbf{R}^n} F(u)\,dx \;:\; u \in D^{1,2}(\mathbf{R}^n)\,,\; \int_{\mathbf{R}^n} |\nabla u|^2 dx \le 1\right\}. \qquad (17)$$

**Lemma 4** *1.* $S_\varepsilon^F(\Omega) \le S^F$ *for every* $\varepsilon > 0$ *and for every* $\Omega \subseteq \mathbf{R}^n$*;*
*2. The following Generalized Sobolev Inequality holds*

$$\int_\Omega F(u)\,dx \le S^F \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{2^*}{2}} \qquad \forall\, u \in D^{1,2}(\Omega)\,;$$

*3.* $S^F \ge F_0 S^*$*.*

*Remark 7.* The above lemma is based on the following scaling argument: if $u \in D^{1,2}(\Omega)$ and $s > 0$, define

$$u^s(x) := u\left(\frac{x}{s}\right) \in D^{1,2}(s\Omega)\,.$$

Then we have

$$\int_{s\Omega} F(u^s)\,dx = s^n \int_\Omega F(u)\,dx \quad \text{and} \quad \int_{s\Omega} |\nabla u^s|^2\,dx = s^{n-2} \int_\Omega |\nabla u|^2\,dx\,.$$

In particular if we choose $s = \|\nabla u\|_{L^2}^{\frac{-2}{n-2}}$ then we also obtain $\int_{s\Omega} |\nabla u^s|^2 dx = 1$. Taking $u^s$ as a competitor for (17), one get the generalized Sobolev inequality.

By a cut-off argument the following result can be also proved.

**Proposition 5** $\lim_{\varepsilon\to 0} S_\varepsilon^F = S^F$*.*

The proof of Lemma 4 and Proposition 5 can be found in [13].

### 3.3 Concentration

From now on we will consider only the case of $\Omega$ bounded and we will write $H_0^1(\Omega)$ instead of $D^{1,2}(\Omega)$. Nevertheless most of the result we will present in this lectures have been obtained for general domains, possibly unbounded.

We are interested in the asymptotic behaviour of maximizing sequences for problem (15); i.e., sequences $u_\varepsilon$ such that

$$\varepsilon^{-2^*} \int_\Omega F(u_\varepsilon)\,dx = S_\varepsilon^F + o(1)\,, \qquad (18)$$

and, possibly, in saying something about their optimal profile.

The case of $F \in C^1(\mathbf{R})$ has been studied by P.L. Lions [21] as an application of the concentration-compactness principle. The general case has been considered by M. Flucher and S. Müller in '99 [13] (see also the book of Flucher [10] and the references therein), still in the spirit of to concentration-compactness.

**Theorem 6** ([13], Theorem 3) *Let* $u_\varepsilon$ *be a maximizing sequence (i.e. satisfying (18)), then there exists* $x_0 \in \overline{\Omega}$ *such that :*

*i) The sequence* $u_\varepsilon$ *concentrates at* $x_0$ *in the following sense*

$$\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_{x_0} \quad \textit{and} \quad \frac{|u_\varepsilon|^{2^*}}{\varepsilon^{2^*}} \overset{*}{\rightharpoonup} S^F \delta_{x_0} \qquad \textit{in } \mathcal{M}(\overline{\Omega})\,; \tag{19}$$

*ii) Suppose that* $S^F > \max\{F_0, F_\infty\}$. *Then we identify the optimal profile for the maximizing sequences. Namely there exists a sequence* $x_\varepsilon$ *converging to* $x_0$ *such that the sequence* $w_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} x)$ *is compact in* $D^{1,2}(\mathbf{R}^n)$ *and every cluster point* $w$ *is a solution for* $S^F$*; i.e., it is a maximum for problem (17).*

*Remark 8.* Let us spend a few words about the condition in part ii) of the theorem. This is a natural condition for those who are familiar with the concentration-compactness approach. It guarantees the existence of a ground state and no concentration of optimal sequences for problem (17). In particular it gives a precise rate of concentration for maximizing sequences of the scaled problem $S_\varepsilon^F$ (we will see that this condition can be slightly relaxed).

It is actually easy to see that $S^F \geq \max\{F_0, F_\infty\}$ is always true. It is enough to take an optimal function for $S^*$, e.g. $u_1$, and define

$$u^s(x) = s^{-\frac{n}{2^*}} u_1\left(\frac{x}{s}\right) .$$

Then from the generalized Sobolev inequality we obtain $S^F \geq F_0 S^*$ and $S^F \geq F_\infty S^*$ taking the limit as $s \to \infty$ and $s \to 0$ respectively. The idea is that the strict inequality, in applying the concentration-compactness principle to the sequence $w_\varepsilon$, is sufficient to rule out vanishing and concentration.

We will see a proof of the concentration result (part i) of Theorem 6) in terms of the variational convergence introduced by De Giorgi in '75 ([8]), the $\Gamma$-convergence.

### 3.4 $\Gamma^+$-convergence

We are considering maximum problems, so the natural variational convergence is $\Gamma^+$-convergence. Since in the literature it is mainly $\Gamma^-$-convergence, the suitable convergence for minimum problems, that is used, we recall quickly the definition and the main properties of $\Gamma^+$-convergence.

In what follows $X$ will be a metric space and $\tau$ will denote its topology.

**Definition 7** *Let* $\mathcal{F}_\varepsilon : X \to \overline{\mathbf{R}}$*, with* $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$*, be a family of functionals. We say that the sequence* $\mathcal{F}_\varepsilon$ $\Gamma^+$-converges *to the functional* $\mathcal{F} : X \to \overline{\mathbf{R}}$ *with respect to* $\tau$*,*

$$\mathcal{F}_\varepsilon \overset{\Gamma^+(X)}{\longrightarrow} \mathcal{F} ,$$

*if the following two properties are satisfied* *i) for every sequence* $x_\varepsilon \xrightarrow{\tau} x$ *we have that*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \le \mathcal{F}(x)\,;$$

*ii) for every* $x \in X$*, there exists a sequence* $x_\varepsilon$*, such that* $x_\varepsilon \xrightarrow{\tau} x$ *and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \ge \mathcal{F}(x)\,;$$

We usually refer to condition i) as the *$\Gamma^+$-limsup inequality* and to condition ii) as the *existence of a recovery sequence.*

The idea of $\Gamma^+$-convergence is that the functional $\mathcal{F}$ describes the main features of the sequence $\mathcal{F}_\varepsilon$ in terms of maxima; i.e., it gives the best (maximal) behaviour of the sequences $\mathcal{F}_\varepsilon(x_\varepsilon)$ along maximizing sequences $x_\varepsilon$.

*Remark 9.*

1. The $\Gamma^+$-limit $\mathcal{F}$ is upper-semicontinuous with respect to $\tau$.
2. If the topology $\tau$ is separable, then $\Gamma^+$-convergence is compact.
3. If there exists a compact set $K \subseteq X$ such that

$$\sup_{x\in X} \mathcal{F}_\varepsilon(x) = \sup_{x\in K} \mathcal{F}_\varepsilon(x)\,,$$

then
a) $\lim_{\varepsilon\to 0}\sup_{x\in X} \mathcal{F}_\varepsilon(x) = \sup_{x\in X} \mathcal{F}(x) = \max_{x\in K} \mathcal{F}(x)$;
b) given a maximizing sequence; i.e., any sequence $x_\varepsilon$ such that

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0}\sup_{x\in X} \mathcal{F}_\varepsilon(x)\,,$$

any cluster point $x$ of $x_\varepsilon$ is a maximum point for $\mathcal{F}$.

A rigorous and complete treatment of the $\Gamma^-$-convergence for the case of minimum problems, whose definition is perfectly symmetric to the one we gave for $\Gamma^+$, can be found for instance in [7] or [4].

### 3.5 The concentration result in terms of $\Gamma^+$-convergence

It is convenient to modify the functional, rescaling the functions by $\varepsilon$ and define

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega F(\varepsilon u)\,dx & \text{if } \displaystyle\int_\Omega |\nabla u|^2 dx \le 1 \\ 0 & \text{otherwise in } L^{2^*}(\Omega). \end{cases} \tag{20}$$

The idea is to consider the functionals $\mathcal{F}_\varepsilon$ and find a suitable topology and a limit functional in order to capture the behaviour stated in Theorem 6 by Flucher and Müller.

**Main requirements:**

1) Choose a space $X$, to which possibly extend the functional $\mathcal{F}_\varepsilon$, rich enough to give information on the maxima;
2) Find a topology which is compact on $X$ in order to get convergence of maxima.

In order to clarify our future choice let us go back to the approach of P.L. Lions. Let $u_\varepsilon$ be a sequence satisfying $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq 1$. Define $\mu_\varepsilon = |\nabla u_\varepsilon|^2 dx$ and try to describe the limit of $\varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\, dx$ in terms of the limit of $\mu_\varepsilon$. Since $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq 1$, we can always assume that $u_\varepsilon$ converges weakly in $H^1_0(\Omega)$ to some function $u$ and that $\mu_\varepsilon$ converges weakly$^*$ in $\mathcal{M}(\overline{\Omega})$ to some measure $\mu$.

As above we may decompose $\mu$ as follows

$$\mu = |\nabla u|^2 dx + \sum_{i\in J} \mu_i \delta_{x_i} + \widetilde{\mu}\,, \tag{21}$$

where $\mu_i$ denotes the positive weight of the atom $x_i$ and $\widetilde{\mu}$ is the non-atomic part of $\mu - |\nabla u|^2 dx$, and define

$$\nu_\varepsilon = \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\, dx\,.$$

By the generalized Sobolev inequality we have

$$\nu_\varepsilon(\Omega) \leq S^F \mu_\varepsilon(\Omega)$$

and thus we have that up to a subsequence $\nu_\varepsilon$ converges weakly$^*$ in $\mathcal{M}(\overline{\Omega})$ to some measure $\nu$. On the other hand by Lemma 1 we know that

$$\nu^*_\varepsilon = |u_\varepsilon|^{2^*} dx \stackrel{*}{\rightharpoonup} \nu^* = |u|^{2^*} dx + \sum_{i\in J} \nu^*_i \delta_{x_i}\,;$$

so that by assumption i) on $F$ we get

$$\nu \leq c\,\nu^*\,. \tag{22}$$

This implies that there exists a function $g \in L^1(\Omega)$ and non-negative numbers $\nu_i$ such that

$$\nu = g(x)\, dx + \sum_{i\in J} \nu_i \delta_{x_i}\,. \tag{23}$$

Then it is clear that, in order to describe the behaviour of $\varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\, dx$ the weak limit in $H^1_0(\Omega)$ of $u_\varepsilon$ is not enough, but we need also to take into account "how" it converges weakly, which is detected by the weak$^*$ limit of the measures $\mu_\varepsilon$. This suggests the choice of the space $X(\Omega)$ as

$$X(\Omega) \subset H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega})$$

and precisely, in view of the constraint $\int_\Omega |\nabla u|^2 dx \le 1$, we will choose

$$X(\Omega) = \left\{(u,\mu) \in H_0^1(\Omega) \times \mathcal{M}(\overline{\Omega}) \,:\, \mu \ge |\nabla u|^2 dx\,,\; \mu(\overline{\Omega}) \le 1\right\}. \qquad (24)$$

Consequently the topology $\tau$ will be chosen such that

$$(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu) \qquad \Longleftrightarrow \qquad \begin{cases} u_\varepsilon \rightharpoonup u & \text{in } w - L^{2^*}(\Omega) \\ \mu_\varepsilon \overset{*}{\rightharpoonup} \mu & \text{in } \mathcal{M}(\overline{\Omega}). \end{cases} \qquad (25)$$

*Remark 10.*

1. With this choice of the topology the space $X(\Omega)$ is metric and separable.
2. It is possible to show that all the pairs $(u,\mu) \in X(\Omega)$ can be obtained as a limit with respect to $\tau$ of pairs of the form $(u_\varepsilon, |\nabla u_\varepsilon|^2)$.
3. The topology $\tau$ is compact in $X(\Omega)$, then the $\Gamma^+$-convergence of functionals in this space implies the convergence of maxima.

Now that we have set the right space we have to extend our functional to $X(\Omega)$ and this is done, by a little abuse of notation, in the natural way as follows

$$\mathcal{F}_\varepsilon(u,\mu) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega F(\varepsilon u)\, dx & \text{if } \mu = \displaystyle\int_\Omega |\nabla u|^2 dx, \\ \\ 0 & \text{otherwise in } X(\Omega). \end{cases} \qquad (26)$$

Then we look for a functional $\mathcal{F}$ which is the $\Gamma^+$-limit of $\mathcal{F}_\varepsilon$ in $X(\Omega)$. Clearly this $\Gamma^+$-limit has to take into account both, the absolutely continuous part of $\mu$ and its atomic part.

**Theorem 8** ([2], Theorem 3.1) *There exists a functional $\mathcal{F} : X(\Omega) \to \mathbf{R}$ which is the $\Gamma^+$-limit of $\mathcal{F}_\varepsilon$ in $X(\Omega)$ and it is given by*

$$\mathcal{F}(u,\mu) := F_0 \int_\Omega |u|^{2^*} dx + S^F \sum_{i\in J} (\mu_i)^{\frac{2^*}{2}}\,. \qquad (27)$$

*Remark 11.* By the $\Gamma^+$-convergence result we immediately deduce the concentration result. Indeed, as already observed, we have that

$$\sup_{(u,\mu)\in X(\Omega)} \mathcal{F}_\varepsilon(u,\mu) = S_\varepsilon^F \to \max_{(u,\mu)\in X(\Omega)} \mathcal{F}(u,\mu)\,;$$

and hence, since by Proposition 5 we have that $S_\varepsilon^F \to S^F$, we get $S^F = \max_{(u,\mu)\in X(\Omega)} \mathcal{F}(u,\mu)$. On the other hand by the Sobolev inequality, Lemma 4 and the convexity of the function $|t|^{\frac{2^*}{2}}$ we get

$$
\begin{aligned}
\mathcal{F}(u,\mu) &= F_0 \int_\Omega |u|^{2^*} dx + S^F \sum_{i\in J} (\mu_i)^{\frac{2^*}{2}} \\
\text{(Sobolev inequality)} &\leq F_0 S^* \left( \int_\Omega |\nabla u|^2 dx \right)^{\frac{2^*}{2}} + S^F \sum_{i\in J} (\mu_i)^{\frac{2^*}{2}} \qquad (28)\\
{\scriptstyle (S^* F_0 \leq S^F)} &\leq S^F \left[ \left( \int_\Omega |\nabla u|^2 dx \right)^{\frac{2^*}{2}} + \sum_{i\in J} (\mu_i)^{\frac{2^*}{2}} \right] \\
&\leq S^F \mu(\overline{\Omega}) \leq S^F \, .
\end{aligned}
$$

Since the Sobolev constant is not attained in $\Omega$ the first inequality is strict unless $u = 0$ and the third inequality is strict unless $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$. In other words

$$
\mathcal{F}(u,\mu) = S^F = \max_{X(\Omega)} \mathcal{F} \qquad \Longleftrightarrow \qquad (u,\mu) = (0, \delta_{x_0}) \, ,
$$

which correspond to the concentration of the maximizing sequence at $x_0$.

*Proof (Theorem 8).* We give a detailed proof of the $\Gamma^+$-limsup inequality (condition i) of Definition 7); i.e. we will prove that for every sequence $(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu)$ then

$$
\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) \leq \mathcal{F}(u,\mu) \, . \tag{29}
$$

We can assume $(u_\varepsilon, \mu_\varepsilon) = (u_\varepsilon, |\nabla u_\varepsilon|^2)$ otherwise the proof is trivial. We already know that in this case

$$
\mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) = \nu_\varepsilon(\overline{\Omega}) = \varepsilon^{-2^*} \int_\Omega F(\varepsilon u_\varepsilon) \, dx \to \int_\Omega g \, dx + \sum_{i\in J} \nu_i = \nu(\overline{\Omega}). \tag{30}
$$

We will give the prove in several steps.
**Step 1.** *(Localization of the generalized Sobolev inequality) For every $\delta > 0$ there exists a constant $k(\delta) > 0$ such that if $0 < r < R$ with $\frac{r}{R} \leq k(\delta)$, then for every $x_0 \in \Omega$*

$$
\int_{B_r(x_0)} F(u) \, dx \leq S^F \left( \int_{B_R(x)} |\nabla u|^2 dx + \delta \int_\Omega |\nabla u|^2 dx \right)^{\frac{2^*}{2}} \tag{31}
$$

*for every $u \in H^1_0(\Omega)$.*

In order to see this step let define the function

$$
\varphi^R_r(x) = \max \left\{ \frac{\log |x_0 - x| - \log R}{\log r - \log R}, 1 \right\} .
$$

This function (the $n$-capacitary potential of the ball $B_r$ with respect to $B_R$) has the following property

$$\int_{B_R(x_0)} |\nabla \varphi_r^R|^n dx \to 0 \quad \text{as} \quad \frac{r}{R} \to 0\,.$$

Moreover it is a cut-off function between $B_r(x_0)$ and $B_R(x_0)$; i.e., $\varphi_r^R(x) = 1$ in $B_r(x_0)$ and $\varphi_r^R(x) = 0$ in $\mathbf{R}^n \setminus B_R(x_0)$.

Now by Hölder's inequality and the Sobolev inequality, for any $\beta > 0$ and for any $u \in H_0^1(\Omega)$, we get

$$\begin{aligned}
&\int_{B_R(x_0)} |\nabla(\varphi_r^R u)|^2 dx \\
&\leq \left(1 + \frac{1}{\beta}\right) \int_{B_R(x_0)} |\nabla \varphi_r^R|^2 |u|^2 dx + (1+\beta) \int_{B_R(x_0)} |\nabla u|^2 dx \\
&\leq \left(1 + \frac{1}{\beta}\right) \left(\int_{B_R(x_0)} |\nabla \varphi_r^R|^n dx\right)^{\frac{2}{n}} \left(\int_{B_R(x_0)} |u|^{2^*} dx\right)^{\frac{n-2}{2}} \\
&\quad +(1+\beta) \int_{B_R(x_0)} |\nabla u|^2 dx \\
&\leq \int_{B_R(x_0)} |\nabla u|^2 dx + \left(\beta + \left(1 + \frac{1}{\beta}\right) S^* \left[\int_{B_R(x_0)} |\nabla \varphi_r^R|^n dx\right]^{\frac{2}{n}}\right) \int_\Omega |\nabla u|^2 dx\,.
\end{aligned}$$

Now if $\beta \leq \delta/2$ and the ratio between $r$ and $R$ is small enough we get

$$\int_{B_R(x_0)} |\nabla(\varphi_r^R u)|^2 dx \leq \int_{B_R(x_0)} |\nabla u|^2 dx + \delta \int_\Omega |\nabla u|^2 dx.$$

Then the conclusion follows applying the Generalized Sobolev inequality to the function $\varphi_r^R u$.

**Step 2.** *We now prove that*

$$\nu_i \leq S^F (\mu_i)^{\frac{2^*}{2}}\,. \tag{32}$$

For any $\delta > 0$, for any atom $x_i \in \Omega$, with $i \in J$, and $\frac{r}{R} \leq k(\delta)$ we may apply the localization of the generalized Sobolev inequality (31) to the functions $\varepsilon u_\varepsilon$ and recalling that $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq 1$ we get

$$\varepsilon^{-2^*} \int_{B_r(x_i)} F(\varepsilon u_\varepsilon)\, dx \leq S^F \left(\int_{B_R(x_i)} |\nabla u_\varepsilon|^2 dx + \delta\right)\,.$$

Taking the limit as $\varepsilon \to 0$ we have

$$\nu(B_r(x_i)) \leq S^F (\mu(B_R(x_i)) + \delta)^{\frac{2^*}{2}}\,,$$

thus the conclusion follows taking the limit as $r \to 0$, by the arbitrariness of $R$ and $\delta$.

**Step 3.** *We finally prove that*

$$g \le F_0 |u|^{2^*}. \tag{33}$$

Here the idea in that large values of $u_\varepsilon$ do not contribute to the absolutely continuous part of $\nu$.

Fix an open subset $U$ of $\Omega$ and $\delta > 0$. Let $t_\delta > 0$ be such that

$$F(t) \le (F_0 + \delta)|t|^{2^*} \qquad |t| < t_\delta\,. \tag{34}$$

Since $U$ is open we have that $\nu(U) \le \liminf_{\varepsilon\to 0} \int_U \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx$ and that $\nu^*(\overline{U}) \ge \limsup_{\varepsilon \to 0} \int_U |u_\varepsilon|^{2^*} dx$; hence, we get

$$\begin{aligned} \int_U g\,dx \le \nu(U) &\le \liminf_{\varepsilon\to 0} \int_U \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx \\ &\le \limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon| < t_\delta\}} \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx \\ &\quad + \limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon| \ge t_\delta\}} \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx \\ &\le (F_0+\delta)\nu^*(\overline{U}) + c \limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon| \ge t_\delta\}} |u_\varepsilon|^{2^*} dx\,, \end{aligned} \tag{35}$$

where for the last inequality we used (34) and the growth condition for $F$. Now, by Lemma 1 and the fact that $\varepsilon u_\varepsilon$ converges to zero in measure we obtain

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon| \ge t_\delta\}} |u_\varepsilon|^{2^*} dx \\ &\quad \le 2^{2^*} \limsup_{\varepsilon\to 0} \left( \int_{U\cap\{|\varepsilon u| \ge t_\delta\}} |u|^{2^*} dx + \int_U |u_\varepsilon - u|^{2^*} dx \right) \\ &\quad = 2^{2^*} \limsup_{\varepsilon\to 0} \int_U |u_\varepsilon - u|^{2^*} dx = \sum_{x_i \in U} \nu_i^* \end{aligned}$$

This together with (35) implies

$$g dx \le F_0 |u|^{2^*} dx + \sum_{i\in J} (F_0 + 2^{2^*} c)\nu_i^* \delta_{x_i}$$

which gives (33). Then the prove of the $\Gamma^+$-limsup inequality is completed.

As for the prove of the existence of a recovery sequence we just sketch it and we refer to [2] for the details. The main steps are the following:

**Step I.** In the case $(u,\mu) = (u, |\nabla u|^2 + \widetilde{\mu})$, any sequence such that $(u_\varepsilon, |\nabla u_\varepsilon|^2)$ converges to $(u, |\nabla u|^2 + \widetilde{\mu})$ in the $\tau$ topology will do the job. Indeed, by

Lemma 1, we know that $u_\varepsilon$ converges strongly in $L^{2^*}$ to $u$, and this, together with the upper semi-continuity of $F$ and the definition of $F_0$, implies that

$$\liminf_{\varepsilon\to 0}\int_\Omega \varepsilon^{-2^*}F(\varepsilon u_\varepsilon)\,dx \geq \int_\Omega F_0|u|^{2^*}\,dx\,.$$

**Step II.** By a localization argument we prove that the $\Gamma^+$-limit exists in every pair $(0,\delta_x)$, with $x\in\Omega$, and coincides with $S^F=\mathcal{F}(0,\delta_x)$. In particular this implies the existence of the recovery sequence for such class of pairs.

**Step III.** All the pairs of the form $(0,\sum_{i\in J}\mu_i\delta_{x_i})$, with $x_i\in\Omega$, are obtained by scaling the recovery sequences for the single atoms in a small ball around the atoms.

**Step IV.** The general case is recovered by a density lemma which permits to glue the different contributions described above.

## 4 Identification of the concentration point

The next natural question, that we will address in this section, is the following: is there a special point of $\Omega$ which is "preferred" for the concentration? For instance, if problem (15) has a maximum for any $\varepsilon$, does the shape of the domain $\Omega$ influence the concentration of the sequence of maxima?

If we look at the example of the capacity it is clear that in order to maximize the volume for fixed capacity the optimal set has to stay "far" from the boundary. This "far" has to be understood in the sense of potential theory. The object that will play a crucial role to make this concept precise is the Green's function for the Dirichlet problem in $\Omega$ with the Laplacian.

*Example 2.* Let us briefly explicitly show that in the case of the *Volume functional* (see problem (14)), with $\Omega=B_R(0)$, the concentration of optimal sets is at the origin. Indeed let us denote by $\rho_\varepsilon$ the radius such that $\mathrm{cap}(B_{\rho_\varepsilon}(0),B_R(0))=\varepsilon^2$ and assume that $A_\varepsilon$ is a set for which the maximum for $S^V_\varepsilon(B_R(0))$ is achieved. Let $u_\varepsilon$ be its capacitary potential and $u^*_\varepsilon$ be its radial symmetrization. In particular we have that $u^*_\varepsilon$ is the potential of the set

$$A^*_\varepsilon=\{u^*_\varepsilon\geq 1\}\,.$$

Moreover by symmetrization we also know that

$$\int_{B_R(0)}|\nabla u^*_\varepsilon|^2dx \leq \int_{B_R(0)}|\nabla u_\varepsilon|^2dx=\mathrm{cap}(A_\varepsilon,B_R(0))=\varepsilon^2$$

and the inequality is strict unless $u_\varepsilon$ is itself radial. Thus if $A_\varepsilon\neq B_{\rho_\varepsilon}(0)$ and denoting by $\rho^*_\varepsilon$ the radius of $A^*_\varepsilon$, then we have

$$\mathrm{cap}(B_{\rho_\varepsilon^*}(0), B_R(0)) < \varepsilon^2 \,,$$

and hence $\rho_\varepsilon^* < \rho_\varepsilon$, which contradicts the maximality of $A_\varepsilon$. In conclusion, recalling that $\mathrm{cap}(B_{\rho_\varepsilon}(0), B_R(0)) = \frac{1}{K(\rho_\varepsilon)-K(R)}$, we proved that the optimal sets are given by

$$B_{\rho_\varepsilon}(0) = \left\{ K(|x|) - K(R) > \varepsilon^2 \right\} \,, \tag{36}$$

and hence they concentrate at the origin.

In the example above we wrote the optimal sets in the form (36) in order to underline the fact that they are given by super-level sets of the Green's function of $-\Delta$ in $B_R(0)$ and with singularity at the origin. In fact in this case the Green's function is given by

$$G^0_{B_R(0)}(x) = K(|x|) - K(R) \,.$$

We will soon see that this is a general fact; we can construct optimal sets for problem $S^V_\varepsilon(\Omega)$ using the super-level sets of the Green's function in $\Omega$ and this is related with the remarkable fact that the rescaled potentials of concentrating sets converges to the Green's function.

### 4.1 The Green's function and the Robin function

Let us recall the definition and the main properties of the Green's function. Assume $\Omega$ be a bounded set with regular boundary (e.g. satisfying the property of the external ball or, more precisely, regular in the sense of Wiener).

**Definition 9** *The* Green's function $G^{x_0}_\Omega(x)$ *for the Dirichlet problem, with the Laplace operator in the domain* $\Omega$ *and singularity* $x_0$*, is the function satisfying*

$$G^{x_0}_\Omega \in H^1(\Omega \setminus B_\rho(x_0)) \cap W^{1,p}_0(\Omega) \qquad \forall\ \rho > 0 \ \ \textit{and} \ \ \forall\ p < \frac{n}{n-1}$$

*and it is a solution in the sense of distribution of the following problem*

$$\begin{cases} -\Delta G^{x_0}_\Omega = \delta_{x_0} & \textit{in } \Omega \\ G^{x_0}_\Omega = 0 & \textit{on } \partial\Omega. \end{cases} \tag{37}$$

The Green's function depends on $\Omega$ through its *regular part*. Actually we can rewrite it as

$$G^{x_0}_\Omega(x) = K(|x - x_0|) - H_\Omega(x_0, x) \,, \tag{38}$$

where $K(|\cdot - x_0|)$ is the fundamental solution and $H_\Omega(x_0, \cdot)$, the regular part, is the solution in the sense of $H^1$ of the following Dirichlet problem

$$\begin{cases} -\Delta H_\Omega(x_0, \cdot) = 0 & \text{in } \Omega \\ G^{x_0}_\Omega(x) = K(|x - x_0|) & \text{if } x \in \partial\Omega. \end{cases} \tag{39}$$

In order to make the ansatz suggested by Example 2 more precise let us fix $x_0 \in \Omega$ and let us consider the set $A_\varepsilon = \{G_\Omega^{x_0} > \varepsilon^{-2}\}$. First we compute $\mathrm{cap}(A_\varepsilon, \Omega)$. Since $G_\Omega^{x_0}$ is harmonic in $\Omega \setminus \{x_0\}$ and it is zero on the boundary of $\Omega$, the potential $u_\varepsilon$ of $A_\varepsilon$ is given by

$$u_\varepsilon = \varepsilon^2 \left(G_\Omega^{x_0} \wedge \varepsilon^{-2}\right) \in W_0^{1,\infty}(\Omega)\,.$$

Then

$$\begin{aligned}\mathrm{cap}(A_\varepsilon, \Omega) &= \int_\Omega |\nabla u_\varepsilon|^2 dx = \varepsilon^4 \int_{\Omega \setminus A_\varepsilon} |\nabla G_\Omega^{x_0}|^2 dx \\ &= \varepsilon^4 \int_\Omega \nabla G_\Omega^{x_0} \nabla (G_\Omega^{x_0} \wedge \varepsilon^{-2})\, dx = \varepsilon^2\,;\end{aligned}$$

i.e., $A_\varepsilon$ is a good competitor for problem $S_\varepsilon^V(\Omega)$. Let us see now heuristically how the volume of the set $A_\varepsilon$ depends on $x_0$. Note that $A_\varepsilon = \{y \in \Omega : K(|x_0 - y|) - H_\Omega(x_0, y) > \varepsilon^{-2}\}$ is contained in a small ball centered in $x_0$. Indeed $H_\Omega(x_0, \cdot)$ is harmonic, and hence bounded, and $K(|x_0 - \cdot|)$ is radial around $x_0$. Moreover, by the uniform continuity of $H_\Omega(x_0, \cdot)$ we get that

$$H_\Omega(x_0, y) = H_\Omega(x_0, x_0) + o(1) \quad \text{as} \quad |x_0 - y| \to 0\,. \tag{40}$$

Then we can rewrite

$$\begin{aligned}A_\varepsilon &= \left\{\gamma_n |x_0 - y|^{2-n} > \varepsilon^{-2} + H(x_0, y)\right\} \\ &= \left\{|x_0 - y| < \left[\frac{\gamma_n}{\varepsilon^{-2} + H(x_0, y)}\right]^{\frac{1}{n-2}}\right\} \\ &= \left\{|x_0 - y| < \varepsilon^{\frac{2}{n-2}} \left[\frac{\gamma_n}{1 + \varepsilon^2 (H(x_0, x_0) + o(1))}\right]^{\frac{1}{n-2}}\right\}.\end{aligned}$$

This, recalling that $S^V = \omega_n \gamma_n^{\frac{n}{n-2}}$, implies that

$$\begin{aligned}|A_\varepsilon| &= S^V \varepsilon^{2^*} \left[\frac{1}{1 + \varepsilon^2 H(x_0, x_0) + o(\varepsilon^2)}\right]^{\frac{n}{n-2}} \\ &= S^V \varepsilon^{2^*} \left[1 - \frac{n}{n-2} \varepsilon^2 H(x_0, x_0) + o(\varepsilon^2)\right].\end{aligned}$$

*Remark 12.* In the computation above, the quantity which determines the volume of $A_\varepsilon$, when $x_0$ varies in $\Omega$ is the regular part of the Green's function, more precisely $H_\Omega(x_0, x_0)$.

**Definition 10** *The diagonal of the regular part of the Green's function is called the* Robin function *of $\Omega$; i.e.,*

$$\tau_\Omega(x) := H_\Omega(x, x) .$$

*We call the harmonic radius of $\Omega$ at $x$, the positive number $\rho_\Omega(x)$ such that*

$$K(\rho_\Omega(x)) = \tau_\Omega(x) .$$

*The points of $\Omega$ where $\tau(x)$ attains its minimum (the maxima for $\rho_\Omega(x)$) are called the harmonic centers of $\Omega$.*

*Remark 13.* The computation that we did before shows that among all the super-level sets of the Green's function, with given capacity, the biggest are those corresponding to the singularity in the harmonic centers of $\Omega$; and hence which concentrates at this points.

In the case of a ball $B_R = B_R(0)$ the Robin function and the harmonic radius can be computed explicitly and they are given by

$$\tau_{B_R}(x) = \gamma_n \left| R - \frac{|x|^2}{R} \right|^{2-n} \quad \text{and} \quad \rho_{B_R}(x) = R - \frac{|x|^2}{R} .$$

Hence $\tau_{B_R}$ attains its minimum at the origin and tends to infinity when $x$ approaches the boundary.

Note that in general the harmonic radius of a domain $\Omega$ at a point $x$ is the radius of the ball whose corresponding Robin function evaluated at the origin agrees with $\tau_\Omega(x)$.

*Remark 14.*

1. In general, if the boundary of $\Omega$ is regular then the Robin function $\tau_\Omega(x)$ tends to infinity as $x$ approaches the boundary. Indeed in this case the regular part of the Green's function, $H_\Omega(x, \cdot)$, attains the boundary condition continuously. We will see that we can extend all these notions to the case of general domains (possibly irregular) and that in some cases this property may be false.
2. In the case of the ball the harmonic center is unique and coincides with the center of the ball. It has been proved by Cardaliaguet and Tahraoni [6] that for any bounded convex domain there exists only one harmonic center.
3. The role of the Green's function for similar problems involving the critical Sobolev exponent was first conjectured by Brezis and Pelletier [5]. In particular the conjecture says that the solutions of the following problem

$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega , \end{cases} \tag{41}$$

with $p < 2^*$, concentrate at critical points of the Robin function of $\Omega$ as $p \to 2^*$. Namely the points $x_p$ where a solution $u_p$ of problem (41) attains the maximum, converge to a critical point of $\tau_\Omega(x)$. This conjecture has been then proved by Rey [22] and Han [18] independently. Later, Flucher and Wei [15] proved that the variational solutions; i.e., the maximizers of

$$S_p(\Omega) = \sup \left\{ \frac{\displaystyle\int_\Omega |u|^p dx}{\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)^{\frac{p}{2}}} \ : \ u \in H_0^1(\Omega) \ \ u \neq 0 \right\} ,$$

concentrate at the harmonic center of $\Omega$.

In view of the example of the capacity, we expect for our general variational problem $S_\varepsilon^F(\Omega)$ a result similar to that described in Remark 14 (3).

We know that for any maximizing sequence $u_\varepsilon$, namely $\mathcal{F}_\varepsilon(u_\varepsilon) = S_\varepsilon^F(\Omega) + o(1)$, there exist a subsequence (still denoted by $u_\varepsilon$) and a point $x_0 \in \overline{\Omega}$ such that $u_\varepsilon$ concentrates at $x_0$. On the other hand for any $x_0 \in \overline{\Omega}$ we can construct a maximizing sequence which concentrates at $x_0$.

Now the question is the following: if we look at maximizing sequences which converge faster, is the concentration point determined by the shape of the domain $\Omega$? This problem has been considered in [11] by means of an asymptotic expansion of the energy. The same result can be stated in terms of $\Gamma^+$-convergence (see [2]). In the latter approach a way to select among maximizing sequences is to consider a first order expansion in $\Gamma^+$-convergence (a sort of Taylor expansion in $\Gamma^+$-convergence).

### 4.2 Asymptotic expansion in $\Gamma^+$-convergence

Assume that a sequence of functionals $\mathcal{F}_\varepsilon$, defined in a metric space $X$, $\Gamma^+$-converges with respect to the topology $\tau$ to some functional $\mathcal{F} : X \to \overline{\mathbf{R}}$. Suppose now that, following an ansatz, we know that

$$\sup_X \mathcal{F}_\varepsilon = \max_X \mathcal{F} + O(\lambda_\varepsilon)$$

for some $\lambda_\varepsilon = o(1)$. We then consider the following functional

$$\mathcal{F}_\varepsilon^1(x) = \frac{\mathcal{F}_\varepsilon - \max_X \mathcal{F}}{\lambda_\varepsilon} .$$

If the $\Gamma^+$-limit $\mathcal{F}^1$ of $\mathcal{F}_\varepsilon^1$ exists, then it clearly will be finite at most on the maxima of $\mathcal{F}$. Furthermore under the usual compactness condition for the topology, we deduce that

$$\lim_{\varepsilon \to 0} \sup_X \mathcal{F}_\varepsilon^1 = \lim_{\varepsilon \to 0} \frac{\sup_X \mathcal{F}_\varepsilon - \max_X \mathcal{F}}{\lambda_\varepsilon} = \max_X \mathcal{F}^1 , \qquad (42)$$

which gives an asymptotic expansion for the suprema of $\mathcal{F}_\varepsilon$; i.e.,

$$\sup_X \mathcal{F}_\varepsilon = \max_X \mathcal{F} + \lambda_\varepsilon \max_X \mathcal{F}^1 + o(\lambda_\varepsilon)\,. \tag{43}$$

In addition we have that any maximizing sequence $x_\varepsilon$ for $\mathcal{F}^1_\varepsilon$; i.e., such that

$$\mathcal{F}^1_\varepsilon(x_\varepsilon) = \sup_X \mathcal{F}^1_\varepsilon + o(1)\,,$$

converges, up to a subsequence, to a maximum of $\mathcal{F}^1$. In particular such a maximizing sequence will also satisfy

$$\mathcal{F}_\varepsilon(x_\varepsilon) = \sup_X \mathcal{F}_\varepsilon + o(\lambda_\varepsilon)\,. \tag{44}$$

In this sense the first order $\Gamma^+$-limit selects, among all maximizing sequences for $\mathcal{F}_\varepsilon$, those which converge faster.

### 4.3 The result

In our case the scaling suggested by the computation for the super-level set of the Green's function is $\lambda_\varepsilon = \varepsilon^2$. Thus our next goal is to compute the $\Gamma^+$-limit of the following functionals

$$\mathcal{F}^1_\varepsilon(u,\mu) = \frac{\mathcal{F}_\varepsilon(u,\mu) - S^F}{\varepsilon^2}\,. \tag{45}$$

In order to obtain a non trivial limit for $\mathcal{F}^1_\varepsilon$ we have to assume an additional condition. Indeed we cannot aspect in general a precise rate of convergence of maximizing sequences. For instance in the critical case $F(t) = |t|^{2^*}$, we may have optimal sequences converging to $S^*$ with any rate. We then need an assumption which forbids this scaling invariance effect. As we already briefly mentioned a sufficient condition is given by

$$S^F > \max\{F_0, F_\infty\} S^*\,. \tag{46}$$

Under this condition we know that there exists a ground state $w$ for $S^F$. The qualitative behaviour of such a solution has been stated by Flucher and Müller in [12]. Among other things they prove that there exists a point $x_0 \in \mathbf{R}^n$ and a ball $B_{R_0}(x_0)$ such that $w$ is radial around $x_0$ outside this ball. More precisely they find a constant $W_\infty$ such that

$$w(x) = W_\infty K(|x-x_0|)(1 + o(r^{-2})) \qquad \forall\, x \,:\, r = |x - x_0| \geq R_0; \tag{47}$$

in other words $w$ is asymptotically proportional to the fundamental solution. The constant $W_\infty$ is given by

$$W^2_\infty = \frac{2(n-1)}{n\, S^F} \int_{\mathbf{R}^n} \frac{F(w(x))}{K(|x|)}\, dx.$$

Set now $w_\infty > 0$ such that

$$w_\infty^2 := \frac{2(n-1)}{n\, S^F} \inf \left\{ \liminf_{k\to\infty} \int_{\mathbf{R}^n} \frac{F(w_k(x))}{K(|x|)}\, dx \right\}, \tag{48}$$

where the infimum is taken among radial maximizing sequences for $S^F$. By using the concentration compactness principle it is shown in [11] that condition (46) implies

$$0 < w_\infty < +\infty\,. \tag{49}$$

We now are ready to state the theorem on the identification of the concentration point in terms of $\Gamma^+$-convergence.

**Theorem 11** ([11] and [2]) *Under condition (46) we have that there exists the $\Gamma^+$-limit $\mathcal{F}^1$ of $\mathcal{F}^1_\varepsilon$ defined by (45) in the space $X(\Omega) = H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega})$ and it is given by*

$$\mathcal{F}^1(u,\mu) = \begin{cases} -\dfrac{n}{n-2} S^F\, w_\infty\, \tau_\Omega(x_0) & \text{if } (u,\mu) = (0,\delta_{x_0}) \\ \\ -\infty & \text{otherwise.} \end{cases}$$

As a consequence of this $\Gamma^+$-convergence result we can deduce the result of concentration for sequences of *almost maximizers*; i.e., satisfying

$$\varepsilon^{-2^*} \int_\Omega F(\varepsilon u_\varepsilon)\, dx = S^F_\varepsilon(\Omega) + o(\varepsilon^2)\,. \tag{50}$$

**Theorem 12** *Under condition (46), all sequences of almost maximizers, in the sense of (50), up to a subsequence, concentrate at a harmonic center of $\Omega$.*

*Proof.* Theorem 11 implies that the sequence $u_\varepsilon$, being a maximizing sequence for $\mathcal{F}^1_\varepsilon$, up to a subsequence, must concentrate at a point $x_0 \in \overline{\Omega}$ which satisfies

$$\mathcal{F}^1(0,\delta_{x_0}) = \max_{X(\Omega)} \mathcal{F}^1(u,\mu)\,,$$

and this implies that $\tau_\Omega(x_0) = \min_\Omega \tau_\Omega$; hence we deduce the concentration at a harmonic center of $\Omega$.

In order to obtain the result above, condition (46) can be relaxed assuming directly condition (49). More details about the importance of condition (49) can be found in [11]. Here we will consider in detail only the proof of Theorem 11 in the case of the Volume problem in (14). In this case we already saw that the solution for $S^V$ is given by

$$w(x) = K(|x|) \wedge 1\,,$$

and hence $w_\infty = 1$.

In the following we will prove Theorem 11 in the capacity case. Namely we will prove

(i) (Existence of the recovery sequence) For every $x_0 \in \Omega$ there exists a sequence of sets $A_\varepsilon$ which concentrates at $x_0$ such that $\mathrm{cap}(A_\varepsilon, \Omega) = \varepsilon^2$ and

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon^{-2^*}|A_\varepsilon| - S^V}{\varepsilon^2} \geq -S^V \frac{n}{n-2} \tau_\Omega(x_0)$$

(ii) ($\Gamma^+$-limsup inequality) For every sequence of sets $A_\varepsilon$ which concentrates at some $x_0$ and satisfies $\mathrm{cap}(A_\varepsilon, \Omega) \leq \varepsilon^2$, we have

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon^{-2^*}|A_\varepsilon| - S^V}{\varepsilon^2} \leq -S^V \frac{n}{n-2} \tau_\Omega(x_0) .$$

Here with concentration of the sets $A_\varepsilon$ at $x_0$, we mean

$$\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_{x_0} ,$$

where $u_\varepsilon$ denotes the capacitary potential of $A_\varepsilon$, or equivalently the pair $(u_\varepsilon/\varepsilon, |\nabla u_\varepsilon|^2/\varepsilon^2)$ converges to $(0, \delta_{x_0})$ in the topology of $X(\Omega)$.

*Proof (i).* The existence of the recovery sequence essentially has been already proved when we computed the volume of the super-level sets of the Green's function. Thus the recovery sequence is given by $A_\varepsilon = \{G_\Omega^{x_0} > \varepsilon^{-2}\}$ and we have

$$|A_\varepsilon| \geq \varepsilon^{2^*} S^V \left(1 - \frac{n}{n-2} \tau_\Omega(x_0)\varepsilon^2 + o(\varepsilon^2)\right) .$$

The proof of the $\Gamma^+$-limsup inequality is based on an asymptotic formula for the capacity of the small sets. The crucial lemma is the following.

**Lemma 13** *Let $x_0 \in \Omega$ and let $A_\varepsilon$ be a sequence of subsets of $\Omega$, with $|A_\varepsilon| > 0$, such that*

$$\frac{|\nabla u_\varepsilon|^2}{\mathrm{cap}(A_\varepsilon, \Omega)} \overset{*}{\rightharpoonup} \delta_{x_0} ,$$

*where $u_\varepsilon$ is the corresponding capacitary potential, then*

$$\liminf_{\varepsilon \to 0} \frac{1}{\mathrm{cap}(A_\varepsilon^*, \mathbf{R}^n)} + \frac{1}{\mathrm{cap}(A_\varepsilon, \Omega)} \geq \tau_\Omega(x_0) . \tag{51}$$

*Remark 15.* The assumption of concentration for the sets $A_\varepsilon$ given in the lemma above can be given in the following weaker form

$$\frac{\chi_{A_\varepsilon}}{|A_\varepsilon|} \overset{*}{\rightharpoonup} \delta_{x_0}$$

(which is actually what we need for the proof of the theorem in the general case).

A complete proof of Lemma 13 can be found in [11], Lemma 16. Let us see now how we obtain the proof of the $\Gamma^+$-limsup inequality using the lemma.

*Proof.* (ii) Without loss of generality we may assume that $A_\varepsilon$ satisfies

$$\limsup_{\varepsilon\to 0} \frac{\mathrm{cap}(A_\varepsilon, \Omega)}{\varepsilon^2} = 1 .$$

Otherwise the sequence $A_\varepsilon$ will not be a maximizing sequence and the $\Gamma^+$-limsup would be trivially satisfied. Then we can apply Lemma 13. By symmetrization we have

$$|A_\varepsilon^*| = |B_{\rho_\varepsilon}| = |A_\varepsilon| \qquad \text{with} \;\; \rho_\varepsilon = \left(\frac{|A_\varepsilon|}{\omega_n}\right)^{\frac{1}{n}}$$

and hence

$$\mathrm{cap}(A_\varepsilon^*, \mathbf{R}^n) = \frac{1}{K(\rho_\varepsilon)} = \frac{1}{\gamma_n}\left(\frac{|A_\varepsilon|}{\omega_n}\right)^{\frac{n-2}{n}} = \left(\frac{|A_\varepsilon|}{S^V}\right)^{\frac{n-2}{n}} .$$

Thus Lemma 13 implies

$$\left(\frac{S^V}{|A_\varepsilon|}\right)^{\frac{n-2}{n}} - \frac{1}{\varepsilon^2} \geq \tau_\Omega(x_0) + o(1) .$$

Then

$$\frac{|A_\varepsilon|}{\varepsilon^{2^*}} \leq \frac{S^V}{[1 + (\tau_\Omega(x_0) + o(1))\varepsilon^2]^{\frac{n}{n-2}}} = S^V\left(1 - \frac{n}{n-2}(\tau_\Omega(x_0) + o(1))\varepsilon^2\right) ,$$

which gives the $\Gamma^+$-limisup inequality.

In the proof of the identification of concentration points for the case of the Volume Functional, Lemma 13 plays a crucial role. It measures the contribution of the boundary of $\Omega$ in the computation of the capacity with respect to $\Omega$. We will not give the proof of Theorem 11 in the general case. The proof of the $\Gamma^+$-limsup inequality uses similar ideas, but it requires a fine analysis of the asymptotic behaviour of a maximizing sequence and the corresponding super-level sets. We just give a few hints for the construction of the recovery sequence for the general case. This indeed gives us the occasion to recall a rearrangement technique which is one of the main tools to get lower bounds for this kind of problems. This is the classical technique of *harmonic transplantation*. Introduced by Hersch in '69 ([20]) it provides information somehow complementary to the information obtained by radial rearrangement.

### 4.4 Harmonic transplantation

**Definition 14** *Denote by* $G_B^0$ *the Green's function of the ball* $B = B_R(0)$ *with singularity at* 0*. Given an arbitrary radial function*

$$U \ : \ B_R(0) \to \mathbf{R}$$

*we can write it as*

$$U = \varphi \circ G_B^0 \,,$$

*for a suitable real function $\varphi$. Now fix $x_0 \in \Omega$. The* harmonic transplantation *of $U$ from $(B_R(0), 0)$ to $(\Omega, x_0)$ is the function*

$$u = \varphi \circ G_\Omega^{x_0} \ : \ \Omega \to \mathbf{R} \,.$$

**Theorem 15** *The harmonic transplantation $u$ of $U$ from $(B_R(0), 0)$ to $(\Omega, x_0)$ satisfies*

1. *The Dirichlet integral is preserved; i.e.,*
$$\int_\Omega |\nabla u|^2 dx = \int_{B_R(0)} |\nabla U|^2 dx \,;$$
2. *If $R = \rho(x_0)$ is the harmonic radius of $x_0$ in $\Omega$, then*
$$\int_\Omega F(u)\, dx \geq \int_{B_R(0)} F(U)\, dx$$
*for every non-negative function $F$;*
3. *If $U_\varepsilon$ is a sequence of radial functions, with $U_\varepsilon = \varphi_\varepsilon \circ G_B^0$, which concentrate at $0$ in the sense that $|\nabla U_\varepsilon|^2 \overset{*}{\rightharpoonup} \delta_0$, then the corresponding harmonic transplantation $u_\varepsilon = \varphi \circ G_\Omega^{x_0}$ from $(B_R, 0)$ to $(\Omega, x_0)$ are concentrates at $x_0$.*

*Proof.* Part 1 is a consequence of the co-area formula. Indeed, by using $G_\Omega^{x_0} \wedge t$ as test function in problem (37) and integrating by parts, it can be easily seen that

$$\int_{\{G_\Omega^{x_0} = t\}} |\nabla G_\Omega^{x_0}|\, d\mathcal{H}^{n-1} = 1 \qquad \forall\, t > 0 \quad \forall\, \Omega \,.$$

Then, using the level sets of the Green's function and recalling that $u = \varphi \circ G_\Omega^{x_0}$ and $U = \varphi \circ G_B^0$, we can write

$$\begin{aligned} \int_\Omega |\nabla u|^2 dx &= \int_\Omega |\varphi'(G_\Omega^{x_0})|^2 |\nabla G_\Omega^{x_0}|^2 dx \\ &= \int_0^{+\infty} |\varphi'(t)|^2 \int_{\{G_\Omega^{x_0} = t\}} |\nabla G_\Omega^{x_0}|\, d\mathcal{H}^{n-1}\, dt = \int_0^{+\infty} |\varphi'(t)|^2 dt \,, \end{aligned}$$

independently of $\Omega$.

Part 2 is a consequence of the following *Mean Value Inequality*: *Fix $x_0 \in \mathbf{R}^n$, $t > 0$ and $\rho > 0$; then among all $\Omega$ such that $x_0 \in \Omega$ and $\rho = \rho_\Omega(x_0)$ the following quantity*

$$\int_{\partial \{G_\Omega^{x_0} > t\}} \frac{1}{|\nabla G_\Omega^{x_0}|}\, dx$$

*is minimal for* $\Omega = B_\rho(x_0)$. This fact can be proved using the isoperimetric inequality on the level sets of the Green's' function and the properties of the harmonic radius (see [3] for a detailed proof). Using this result we then have

$$\int_\Omega F(u)\,dx = \int_0^{+\infty} F(\varphi(t)) \int_{\partial\{G_\Omega^{x_0}>t\}} \frac{1}{|\nabla G_\Omega^{x_0}|}\,dx\,dt \geq \int_{B_{\rho(x_0)}} F(u)\,dx.$$

As we said harmonic transplantation is the basic tool in order to construct a recovery sequence in the general case. To understand how to make use of this tool, let us briefly show the main lines in the construction. Suppose that we are able to construct a recovery sequence in the special case $\Omega = B_R(0)$ with concentration at the center; namely suppose that we have a sequence $U_\varepsilon : B_R(0) \to \mathbf{R}$ of radial functions which concentrates at the origin, with $\int_{B_R} |\nabla U_\varepsilon|^2 dx = 1$ and satisfying

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon^{-2^*}\displaystyle\int_{B_R(0)} F(\varepsilon U_\varepsilon)\,dx - S^F}{\varepsilon^2} \geq -\frac{n}{n-2} w_\infty \tau_{B_R}(0)\,. \tag{52}$$

Now given $\Omega \subseteq \mathbf{R}^n$, $x_0 \in \Omega$ and fix $R = \rho_\Omega(x_0)$ the harmonic radius of $\Omega$ at $x_0$, by harmonic transplantation we get a sequence $u_\varepsilon : \Omega \to \mathbf{R}$, which concentrates at $x_0$, such that

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = 1,$$

and from the form of the Robin function for a ball, the definition of the harmonic radius and (52) satisfies

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{\varepsilon^{-2^*}\displaystyle\int_\Omega F(\varepsilon u_\varepsilon)\,dx - S^F}{\varepsilon^2} &\geq \liminf_{\varepsilon\to 0} \frac{\varepsilon^{-2^*}\displaystyle\int_{B_R(0)} F(\varepsilon U_\varepsilon)\,dx - S^F}{\varepsilon^2} \\
&\geq -\frac{n}{n-2} w_\infty \tau_{B_R}(0) \\
&= -\frac{n}{n-2} w_\infty K(\rho(x_0)) \\
&= -\frac{n}{n-2} w_\infty \tau_\Omega(x_0)\,;
\end{aligned}$$

hence $(u_\varepsilon, |\nabla u_\varepsilon|^2)$ is a recovery sequence for $(0, \delta_{x_0})$.

*Remark 16.* The case $\Omega = B_R(0)$, with concentration in the center can be done by hand, taking into account that the solution $w$ for $S^F$ is radial and behaves as the fundamental solution at $\infty$. Then one can construct $U_\varepsilon$ truncating $w$ and scaling it (see [11] for details).

## 5 Irregular domains

In this last section we will briefly consider the case of domains $\Omega$ with possibly irregular boundary. In order to deal with this case we should be able to define the Robin function and the harmonic radius up to the boundary for possibly irregular domains. Indeed if the domain is irregular in general one can not aspect the harmonic center being attained at an interior point (see Example 3).

With the following example we exhibit a domain whose harmonic center is at the boundary.

*Example 3.* Let $\Omega_0 = B_1(0)$ and let $\tau_{\Omega_0}$ be the corresponding Robin function. The harmonic center for $\Omega_0$ is 0 and $\tau_{\Omega_0}$ is strictly convex. The idea is to construct two symmetric sequences of small balls centered in the points $(\frac{1}{2^n}, 0, \dots, 0)$ and $(-\frac{1}{2^n}, 0, \dots, 0)$ respectively with radii which go to zero, in a way that the set obtained from $\Omega_0$ by subtracting a finite number of symmetric pairs of balls has its unique harmonic center in the origin.

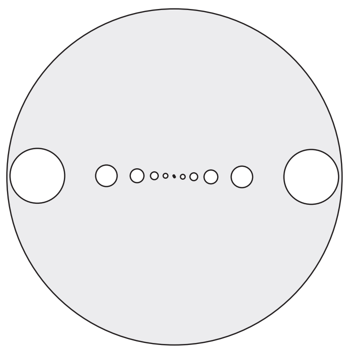

**Fig. 1.** The set $\Omega_n = \Omega_0 \setminus (\cup_{i=1}^n (\overline{B_{\rho_i}(x_i^+)} \cup \overline{B_{\rho_i}(x_i^-)}))$

Let us denote by $x_n^\pm = (\pm\frac{1}{2^n}, 0, \dots, 0)$, $n \in I\!N$ and let $\varepsilon_1 > 0$ be such that $0 < \varepsilon_1 < \min_{\Omega_0 \setminus B_{\frac{1}{4}}(0)} \tau_{\Omega_0} - \min_{B_{\frac{1}{4}}(0)} \tau_{\Omega_0}$. Fix $0 < \alpha < 1/2$, let $\rho_1 > 0$ and denote $\Omega_1 = \Omega_0 \setminus (\overline{B_{\rho_1}(x_1^+)} \cup \overline{B_{\rho_1}(x_1^-)})$. It is easy to check that $\tau_{\Omega_1}$ converges uniformly to $\tau_{\Omega_0}$, as $\rho_1$ tends to zero, in $\Omega_0 \setminus (B_{\rho_1^\alpha}(x_1^+) \cup B_{\rho_1^\alpha}(x_1^-))$ and the same is true for the derivatives. Thus we can choose $\rho_1$ small enough such that $\tau_{\Omega_1}$ is strictly convex on $\Omega_0 \setminus (B_{\rho_1^\alpha}(x_1^+) \cup B_{\rho_1^\alpha}(x_1^-))$, $B_{\rho_1^\alpha}(x_1^\pm) \cap B_{\frac{1}{4}}(x_1^\pm) = \emptyset$ and we have

$$\tau_{\Omega_0}(x) \leq \tau_{\Omega_1}(x) \leq \tau_{\Omega_0}(x) + \frac{\varepsilon_1}{2} \qquad \forall\, x \in \Omega_0 \setminus (B_{\rho_1}(x_1^+) \cup B_{\rho_1}(x_1^-)) .$$

This implies that the harmonic center of $\Omega_1$ is unique, and arguing by symmetry, we conclude that it is in the origin.

By induction we can construct a sequences $\{\rho_n\}$, such that the sets $\Omega_n = \Omega_0 \setminus (\cup_{i=1}^n (\overline{B_{\rho_i}(x_i^+)} \cup \overline{B_{\rho_i}(x_i^-)}))$ have a unique harmonic center at the origin. In particular $\mathrm{dist}(0, \partial\Omega_n) \to 0$ as $n \to \infty$.

**Question**: what happens in domains with irregular boundary? Can we still prove a concentration result like Theorem 4.3?

Note that all the techniques we used in the previous section make use of the fact that the concentration point is an interior point for $\Omega$. Nevertheless the answer to the second question is *yes*, in order to state and prove the concentration result we need a good definition of the Robin function for irregular domains up to the boundary. This can be done in three steps.

**Step 1.** For any $x_0 \in \overline{\Omega}$ we can define the regular part of the Green's function $H_\Omega(x_0, \cdot)$ as the solution in the sense of Perron-Wiener-Brelot of the following Dirichlet problem

$$\begin{cases} \Delta_y H_\Omega(x_0, y) = 0 & \text{in } \Omega, \\ H_\Omega(x_0, y) = K(|x_0 - y|) & \text{on } \partial\Omega; \end{cases} \tag{53}$$

i.e., $H_\Omega(x_0, \cdot)$ is the infimum of all superharmonic functions $u$ such that

$$\liminf_{\substack{z \to y \\ z \in \Omega}} u(z) \geq K(|x_0 - y|)$$

for every $y \in \partial\Omega$ (see [19]). Note that $K(|x_0 - \cdot|)$ is an admissible boundary condition in order to get a unique solution for problem (53).

**Step 2.** We may extend $H(x_0, \cdot)$ to the boundary of $\Omega$ as follows

$$\widetilde{H}_\Omega(x_0, y_0) = \liminf_{\substack{y \to y_0 \\ y \in \Omega}} H(x_0, y)\,.$$

**Step 3.** We now can define the Robin function up to the boundary as

$$\tau_\Omega(x_0) = \widetilde{H}_\Omega(x_0, x_0)\,.$$

With the definition above of $\tau_\Omega$ we can state and prove Theorem 4.3 for any domain, possibly irregular.

*Remark 17.* One could be tempted to define the Robin function up to the boundary simply taking the lower semi-continuous extension of $\tau_\Omega(x) = H_\Omega(x, x)$ with $x \in \Omega$. In dimension $n \geq 5$ one can construct an example which shows that the two procedures do not give the same function; i.e., the Robin function defined by Step 3 can be strictly smaller at the boundary than its lower semi-continuous extension from $\Omega$.

The definition of $\tau_\Omega$ permits also to show that it satisfies the following properties.

**Proposition 16** *a) $\tau_\Omega$ is lower semi-continuous in $\overline{\Omega}$;*
*b) For any $\rho > 0$ and any $x_0 \in \overline{\Omega}$ we denote by $\tau_\rho$ the Robin function corresponding to the domain $\Omega \cup B_\rho(x_0)$. Then we have that $\tau_\rho$ converges increasingly to $\tau_\Omega$ as $\rho \to 0$.*

*Remark 18.* Note that property b) in the proposition above is essential to extend the result to arbitrary domains. The idea is that it permits to consider a boundary point of $\Omega$ as an interior point for a slightly perturbed domain and hence use the same techniques used in the regular case. Indeed first it can be shown that with this definition of $\tau_\Omega$, harmonic transplantation works up to the boundary. Second, it permits to extend Lemma 13 for sets which concentrates at points $x_0$ of the boundary where $\tau_\Omega(x_0) < \infty$. In fact the lemma does not require regularity, but it require $x_0$ to be an interior point. Then it can be applied to the set $\Omega \cup B_\rho(x_0)$ and gives

$$\liminf_{\varepsilon\to 0} \frac{1}{\mathrm{cap}(A^*_\varepsilon, \mathbf{R}^n)} + \frac{1}{\mathrm{cap}(A_\varepsilon, \Omega \cup B_\rho(x_0))} \geq \tau_\rho(x_0)\,.$$

By the fact that $\mathrm{cap}(A_\varepsilon, \Omega \cup B_\rho(x_0)) < \mathrm{cap}(A_\varepsilon, \Omega)$ and the previous proposition we get

$$\liminf_{\varepsilon\to 0} \frac{1}{\mathrm{cap}(A^*_\varepsilon, \mathbf{R}^n)} + \frac{1}{\mathrm{cap}(A_\varepsilon, \Omega)} \geq \tau_\Omega(x_0)\,.$$

*Remark 19.* Proposition 16 shows that $\tau_{\Omega_n}(x)$ constructed in Example 3 converges to the Robin function $\tau_{\Omega_\infty}(x)$ for the set

$$\Omega_\infty = \Omega_0 \setminus \overline{(\cup_{i=1}^\infty (B_{\rho_i}(x_i^+) \cup B_{\rho_i}(x_i^-)))}\,,$$

for every $x \in \overline{\Omega}_\infty$. In particular, since $\{\tau_{\Omega_n}\}$ is an increasing sequence, 0 is the harmonic center of $\Omega_\infty$ and by construction belongs to the boundary of $\Omega_\infty$.

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# Gamma-convergence of gradient flows and applications to Ginzburg-Landau vortex dynamics

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**Summary.** We present in parallel an abstract method of $\Gamma$-convergence of gradient flows, designed to pass to the limit in PDEs which are steepest-descent for functionals which have an asymptotic $\Gamma$-limit energy; together with the application to the Ginzburg-Landau energy. We give schematic proofs of the $\Gamma$-convergence results for Ginzburg-Landau and of the derivation of the dynamical law of vortices through the abstract method.

## 1 Introduction

### 1.1 Presentation of the Ginzburg-Landau model

The Ginzburg-Landau energy was introduced by Ginzburg and Landau in the 50s as a model for superconductivity. It was first a phenomenological theory, but it was later derived (in a certain limit) from the microscopic (quantic) theory of Bardeen-Cooper-Schrieffer. It is now a widely accepted model, which has earned its inventors the Physics Nobel Prize (to Ginzburg, Abrikosov, and in 2003 Ginzburg). Another motivation is the modelling of superfluidity (a phenomenon very close to superfluidity, both mathematically and physically, with a joint Nobel Prize for Legett in 2003) and of Bose-Einstein condensates in rotation (Bose-Einstein condensates were predicted by Bose and Einstein in the early 20th century, and only first realized experimentally in the 90's (it was worth another Nobel Prize...). All these physical phenomena have in common the appearance of *topological vortices*, which are the main object of our study.

Superconductors have this striking feature that "they repel an applied magnetic field" (this is called the Meissner effect). This is true at least when the intensity of the applied field $h_{\mathrm{ex}}$ is not too large; when it becomes larger than a first critical field $H_{c_1}$, then the first *vortices* appear and the magnetic field penetrates through them; when the applied field is further raised, there are more and more vortices, until superconductivity is totally destroyed and the magnetic field completely penetrates the sample. For further reference, we refer to the physics literature, e.g. [24, 7].

The samples are 3D, however, we will consider only the 2D model for simplicity (it already contains most of the important features). The 2D Ginzburg-Landau energy in non-dimensional form is

$$G_\varepsilon(u,A)=\frac{1}{2}\int_\Omega |\nabla_A u|^2+|\text{curl } A-h_{ex}|^2+\frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2. \tag{1}$$

Here $\Omega$ denotes a smooth bounded and *simply connected* domain corresponding to the cross-section of the sample (assuming everything is translation-invariant in the third direction). The function $u:\Omega\to\mathbb{C}$ is called the *order parameter*, $|u(x)|^2\le 1$ indicates the local (normalized) density of superconducting electrons (the "Cooper pairs"). Where $|u(x)|\sim 1$ it is the superconducting phase, where $|u(x)|\sim 0$, it is the normal phase. This order parameter is coupled, in a *gauge-invariant* fashion, to a magnetic potential $A:\Omega\to\mathbb{R}^2$, and the function $h=\text{curl}\,A=\partial_2 A_1-\partial_1 A_2$ is the induced magnetic field in the sample. The real parameter $h_{\text{ex}}$ is the intensity of the external applied magnetic field.

The parameter $1/\varepsilon$ is called the Ginzburg-Landau parameter, it is a dimensionless parameter depending on the material (ratio of two characteristic lengths). When $1/\varepsilon$ is large enough, we are in the category of "type-II" superconductors, when $\varepsilon\to 0$, they are sometimes called "extreme type-II" (or this is also called the "London limit"). This is the asymptotic regime we will be interested in.

## Vortices

Vortices are objects centered at zeros of the order parameter $u$ which carry a nonzero topological degree. Typically, around a vortex centered at a point $x_0$, $u$ "looks like" $u=\rho e^{i\varphi}$ with $\rho(x_0)=0$ and $\rho=f(\frac{|x-x_0|}{\varepsilon})$ where $f(0)=0$ and $f$ tends to 1 as $r\to+\infty$, i.e. its characteristic core size is $\varepsilon$, and

$$\frac{1}{2\pi}\int\frac{\partial\varphi}{\partial\tau}=d\in\mathbb{Z}$$

is an integer, called the *degree of the vortex*. For example $\varphi=d\theta$ where $\theta$ is the polar angle centered at $x_0$ yields a vortex of degree $d$. We have the important relation

$$\text{curl}\,\nabla\varphi=2\pi\sum_i d_i\delta_{a_i}$$

where the $a_i$'s are the centers of the vortices and the $d_i$ their degrees.

**Simplified model (no magnetic coupling)**

A simplified model consists in taking $A = 0$ and $h_{\mathrm{ex}} = 0$, then the energy reduces to

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \tag{2}$$

with still $u : \Omega \to \mathbb{C}$. Critical points of this energy are solutions of

$$-\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2). \tag{3}$$

The first main study of this functional was done by Bethuel-Brezis-Hélein in the book [2]. Since then, a large literature on it has developed. In these notes, for simplicity, we will focus only on this energy (2), however all our results can be extended to the case of (1).

For more reference on (1), and results on its minimizers, their vortices, critical fields, etc, we refer to the monograph [20] and the references therein. In what follows we will often be a little imprecise in the statements for the sake of simplicity, however exact and rigorous corresponding statements can easily be found in the references.

### 1.2 Gamma-convergence

Let us now present the totally independent concept of $\Gamma$-convergence. It was introduced by De Giorgi in the 70s, it served to unify various notions of variational convergence.

The idea is dimension-reduction: when there is a small parameter $\varepsilon \to 0$, reduce the minimization of some original functionals $E_\varepsilon$ to that of a limiting energy $F$, defined on a lower-dimensional space.

A celebrated example of $\Gamma$-convergence was the case of the energy of the "gradient theory of phase-transitions" studied in the 80's by Modica and Mortola (see also Sternberg):

$$M_\varepsilon(u) = \varepsilon\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega (1-u^2)^2 \qquad u : \Omega \to \mathbb{R} \tag{4}$$

that is the same as (2) but for *real-valued* functions. It was established that if for a family $u_\varepsilon$, $M_\varepsilon(u_\varepsilon) \le C$, then, up to extraction of a subsequence, $u_\varepsilon \to u_0$ in $BV(\Omega)$ (the space of functions of bounded variation), with $u_0$ valued in $\{1,-1\}$ and

$$M_\varepsilon \xrightarrow{\Gamma} \frac{8}{3}\text{ per }\gamma = \frac{8}{3}\text{ per }(\partial\{u_0 = 1\}) = \frac{4}{3}\int |Du_0| = \frac{4}{3}\|u_0\|_{BV}$$

where $\gamma$ (typically a codimension 1 object), is the interface between $\{u_0 = 1\}$ and $\{u_0 = -1\}$.

A trick that was used was to write $a^2+b^2 \geq 2ab$ (with equality if $a=b$) hence

$$\varepsilon \int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\int_\Omega (1-u_\varepsilon^2)^2 \geq 2\int |\nabla u_\varepsilon||1-u_\varepsilon^2| \geq 2\int_\Omega \left|\nabla\left(u_\varepsilon - \frac{u_\varepsilon^3}{3}\right)\right|$$

and thus passing to the limit,

$$\underline{\lim}_{\varepsilon\to 0} M_\varepsilon(u_\varepsilon) \geq \frac{8}{3} \text{ per } \gamma.$$

Conversely, given an interface $\gamma$ (a curve if $\Omega \subset \mathbb{R}^2$), one can construct $u_\varepsilon$ such that $M_\varepsilon(u_\varepsilon) \to \frac{8}{3}\text{per}(\gamma)$. This necessitates to paste transversally to $\gamma$ the optimal profile such that $a=b$ above i.e. $\sqrt{\varepsilon}|\nabla u| = \frac{1}{\sqrt{\varepsilon}}\left|1-u^2\right|$, that is

$$u_\varepsilon(x_1,x_2) \simeq \tanh\left(\frac{x_1}{\varepsilon}\right)$$

where $x_1$ is the coordinate in the direction normal to $\gamma$.

**Definition 17 ($\Gamma$-convergence)** *A family of functionals $E_\varepsilon$ (defined on $\mathcal{M}_\varepsilon$) $\Gamma$-converges to a functional $F$ (defined on $\mathcal{N}$) if*

1. *If $E_\varepsilon(u_\varepsilon) \leq C$ then up to extraction of a subsequence, $u_\varepsilon \overset{S}{\rightharpoonup} u \in \mathcal{N}$, and for every $u_\varepsilon \overset{S}{\rightharpoonup} u \in \mathcal{N}$ we have*
$$\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u)$$
2. *For every $u \in \mathcal{N}$, there exists $u_\varepsilon \in \mathcal{M}_\varepsilon \overset{S}{\rightharpoonup} u \in \mathcal{N}$ such that*
$$\overline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \leq F(u)$$

The sense of convergence $S$ is to be specified beforehand. It can be a weak or strong convergence of $u_\varepsilon$, it can also be a convergence of a nonlinear function of $u_\varepsilon$.

In the case of the functional $M_\varepsilon$, one should take $u_\varepsilon \overset{S}{\rightharpoonup} \gamma \Longleftrightarrow u_\varepsilon \to u_0$ in $L^1(\Omega)$ with $Du_0 = \frac{4}{3}\mathcal{H}^{n-1}\lfloor\gamma$ where $\gamma$ denotes a codimension one rectifiable current, and $\mathcal{H}$ the Hausdorff measure.

$\Gamma$-convergence thus requires two conditions: a lower bound, usually obtained via abstract arguments (together with a compactness result), and an upper bound, usually obtained via explicit constructions.

**Proposition 1** *If $E_\varepsilon$ $\Gamma$-converges to $F$ and $u_\varepsilon$ minimizes $E_\varepsilon$ with $E_\varepsilon(u_\varepsilon) \leq C$, then, up to extraction $u_\varepsilon \overset{S}{\rightharpoonup} u$ and $u$ minimizes $F$.*

*Proof.* By 1) of the definition, after extraction, $u_\varepsilon \overset{S}{\rightharpoonup} u$ and $\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u)$. Let us assume that there exists $u_0 \in \mathcal{N}$ such that $F(u_0) < F(u)$, then by 2) of the definition, there exists $v_\varepsilon$ such that $\overline{\lim}_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) \leq F(u_0) < F(u) \leq \underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$. Thus for $\varepsilon$ small enough, we find $E_\varepsilon(v_\varepsilon) < E_\varepsilon(u_\varepsilon)$ contradicting the minimality of $u_\varepsilon$. Hence $u$ must minimize $F$.

In other words "minimizers converge to minimizers". Minimizing $M_\varepsilon$ defined over $H^1(\Omega,\mathbb{R})$ for example reduces to minimizing $F(\gamma) = per(\gamma)$ defined over finite-perimeter sets. It thus achieves a dimension-reduction (since the set of finite-perimeter objects has, somewhat, a lower dimension than $H^1(\Omega,\mathbb{R})$). In general, not much more can be said. For example $u_\varepsilon$ local minimizer of $E_\varepsilon$ *does not imply* $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ local minimizer; or $u_\varepsilon$ critical point of $E_\varepsilon$ *does not imply* $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ critical point of $F$. It is easy to construct finite-dimensional counter-examples.

## 1.3 *Γ*-convergence of Ginzburg-Landau

### Energy lower bound

To obtain a lower bound (and thus a *Γ*-convergence result) for the Ginzburg-Landau functional (2) is more difficult than for $M_\varepsilon$ (the $a^2+b^2 \geq 2ab$ trick doesn't work).

Let us present (formally) some essential ingredients of the analysis of [2]. What is the cost of a radial vortex of degree $d$ of the form $f\left(\frac{r}{\varepsilon}\right)\, e^{i\, d\theta}$? First, formally

$$\frac{1}{2}\int_{B_R} |\nabla u|^2 \geq \frac{1}{2}\int_{R\geq|x|\geq\varepsilon} |f|^2\, \frac{d^2}{r^2}\, r\, dr\, d\theta = \pi d^2 \int_\varepsilon^R \frac{dr}{r} = \pi d^2 \log\frac{R}{\varepsilon} \tag{5}$$

where we have assumed that $f$ is close to 1 for $|x| \geq \varepsilon$. In fact this bound is optimal, at least in the case $d = \pm 1$ as can be seen: if $u \in \mathbb{S}^1$, $u = e^{i\varphi}$, and $|\nabla u| = |\nabla\varphi|$, so

$$\begin{aligned}
\frac{1}{2}\int_{R\geq|x|\geq\varepsilon} |\nabla u|^2 &\geq \frac{1}{2}\int_\varepsilon^R \left(\int_{\partial B_r} \left|\frac{\partial\varphi}{\partial\tau}\right|^2\right) dr \\
&\geq \frac{1}{2}\int_\varepsilon^R \left(\left(\int_{\partial\, B_r} \frac{\partial\varphi}{\partial\tau}\right)^2 \frac{1}{2\pi r}\right) dr \quad \text{(by Cauchy-Schwarz)} \\
&\geq \frac{1}{2}\,\frac{4\pi^2}{2\pi}\int_\varepsilon^R \frac{dr}{r} = \pi\,\log\frac{R}{\varepsilon}
\end{aligned}$$

valid for any degree $\pm1$ vortex (not necessarily radial). Vortices of degree $> 1$ cost more energy than several vortices of degree 1 and are in fact unstable. The cost of $f\left(\frac{r}{\varepsilon}\right)$ imposes the length scale $\varepsilon$, and costs only $O(1)$, which is negligible compared to $\log\frac{1}{\varepsilon}$.

If $u_\varepsilon$ has vortices at points $a_1^\varepsilon, \cdots, a_n^\varepsilon$, of degrees $d_1, \cdots, d_n$, one expects that

$$E_\varepsilon(u_\varepsilon) \geq \pi\left(\sum_i |d_i|\right)\log\frac{1}{\varepsilon}.$$

In fact, this estimate has been made rigorous under certain conditions in [2], and more generally with the "ball construction method" of Sandier/Jerrard (see [17, 11, 20]).

How to trace the vortices? The easiest way is to use the current $\langle iu, \nabla u\rangle$ (or the "superconducting current" $\langle iu, \nabla_A u\rangle$ for the case with magnetic field) where $\langle ., .\rangle$ denotes the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$, i.e. $\langle iu, \nabla u\rangle = (u \times \partial_1 u, u \times \partial_2 u)$ with $\times$ the vector product in $\mathbb{R}^2$. Writing $u = \rho e^{i\varphi}$ we have (at least formally)

$$\langle iu, \nabla u\rangle = \rho^2 \nabla \phi$$

and since $\rho$ is close to 1 on length scales $\varepsilon$, the quantity

$$\text{curl}\,\langle iu, \nabla u\rangle = \text{curl}\,(\rho^2 \nabla\varphi) \simeq \text{curl}\,\nabla\varphi = 2\pi \sum_i d_i \delta_{a_i} \tag{6}$$

can be used to trace the vortices.

This is also called the Jacobian determinant if written (with differential forms) $Ju = \frac{1}{2} d\langle iu, du\rangle = \frac{1}{2}\langle idu, du\rangle = u_{x_1} \times u_{x_2}$. The approximation is justified as a limit as $\varepsilon \to 0$:

**Theorem 1 (see [13, 20])** *Assume $E_\varepsilon(u) \le C|\log\varepsilon|$, then there exists a family of disjoint closed balls $B_i = B(a_i, r_i)$ with $|\log\,\varepsilon|^{-2} \le \sum r_i \le o(1)$ as $\varepsilon \to 0$, such that*

$$\left\{|u| \le \frac{1}{2}\right\} \subset \cup_i B_i \quad \textit{(the $B_i$'s cover the zeroes of $u_\varepsilon$)}$$

$$\frac{1}{2}\int_{\cup_i B_i} |\nabla u_\varepsilon|^2 \ge \pi \sum_i |d_i| \left(\log \frac{\sum_i r_i}{\varepsilon \sum_i |d_i|} - C\right) \qquad d_i = deg\,(u, \partial B_i) \tag{7}$$

$$\|\text{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle - 2\pi \sum_i d_i\, \delta_{a_i}\|_{(C_0^{0,\gamma}(\Omega))^*} \le o(1) \quad \textit{as} \quad \varepsilon \to 0 \tag{8}$$

Combining the upper bound $E_\varepsilon(u) \le C|\log\,\varepsilon|$ and the lower bound (7), we deduce that $\sum_i |d_i| \le C$ for a constant $C$ independent of $\varepsilon$ and thus the number of vortices of nonzero degree remains bounded independently of $\varepsilon$. Thus, if $u_\varepsilon$ is a family of such configurations, once the $a_i^\varepsilon$'s are found, we may extract a subsequence such that $\sum_i d_i \delta_{a_i^\varepsilon} \to \sum_i d_i \delta_{a_i}$ in the weak sense of measures. These fixed points $a_1, \cdots, a_n$ are the limiting vortices. We will sometimes write $u = (a_i, d_i)$ for the limiting points+degrees configurations in $(\Omega \times \mathbb{Z})^n$.

Then the $\Gamma$-convergence result can simply be written

**Theorem 2** *1) Assume $u_\varepsilon$ is such that $\frac{E_\varepsilon(u_\varepsilon)}{|\log\,\varepsilon|} \le C$, then up to extraction*

$$u_\varepsilon \overset{s}{\rightharpoonup} u = (a_i, d_i) \quad \textit{in the sense} \quad \text{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i\, \delta_{a_i} = 2J$$

*and*

$$\varliminf_{\varepsilon\to 0}\frac{E_\varepsilon(u_\varepsilon)}{|\log\,\varepsilon|}\geq \pi\sum_{i=1}^n |d_i| = \|J\|$$

*2) Conversely given any* $(a_i, d_i) \in (\Omega\times\mathbb{Z})^n$*, there exists* $u_\varepsilon$ *such that* $\overline{\lim}_{\varepsilon\to 0}\frac{E_\varepsilon(u_\varepsilon)}{|\log\,\varepsilon|}\leq \pi\sum_{i=1}^n |d_i| = \|J\|$.

This result is not very interesting since it reduces minimizing $E_\varepsilon$ to minimizing the number of points!... It is mostly interesting in higher dimensions. Then, in 3D for example, vortices are not points but vortex-lines, and the Jacobian $Ju_\varepsilon = \frac{1}{2}d(iu_\varepsilon, du_\varepsilon)$ can be seen as a current carried by the vortex-line, converging to a $\pi$ times integer-multiplicity dimension 1 rectifiable current (i.e. line) $J$ and

$$\frac{E_\varepsilon}{|\log\,\varepsilon|}\xrightarrow{\Gamma}\|J\| = \text{length of line (or surface...)}$$

is the lower bound of $\Gamma$-convergence (see [13] and Section 3.1). Thus, $\Gamma$-convergence reduces to minimizing the length of the line, leading to straight lines, a nontrivial problem. In higher dimensions, it leads to codimension 2 minimal currents similarly to $M_\varepsilon$ (see [16, 3]).

In fact, in order for the problem to become interesting in 2D, we need to impose some boundary conditions, for example $u_\varepsilon = g\ \partial\Omega$ with $\deg g\neq 0$ so that there *have* to be vortices, and to look at the next order of the energy in the expansion. This rather arbitrary boundary requirement is in contrast with the case of the full functional (1), for which the natural boundary condition is Neumann, and vortices appear due to the applied magnetic field.

### Renormalized energy

Let us return to lower bounds in order to look for the next order term in the energy (still with formal arguments). Cutting out holes $\cup_i B(a_i,\rho)$ of fixed size $\rho$ around the limiting vortices $a_i$, we may assume that $|u|\sim 1$ in $\Omega\setminus\cup_i B(a_i,\rho) = \Omega_\rho$, and that $u = e^{i\varphi}$, with $\varphi$ a real-valued function, *not* single-valued though (i.e. only defined modulo $2\pi$). Minimizing the energy outside of the holes amounts to solving

$$\min_{\substack{u:\Omega_\rho\to\mathbb{S}^1\\ u=g\text{ on }\partial\Omega\\ deg(u,\partial B(a_i,\rho))=d_i}}\frac{1}{2}\int_{\Omega_\rho}|\nabla u|^2.$$

This is a harmonic map problem, whose solution is given in terms of $\varphi$ by

$$\begin{cases}\Delta\varphi = 0 & \text{in } \Omega_\rho\\ \frac{\partial\varphi}{\partial\tau} & \text{given on } \partial\Omega\\ \int_{\partial B(a_i,\rho)}\frac{\partial\varphi}{\partial\tau} = 2\pi d_i. & \end{cases}$$

and in terms of the harmonic conjugate $\Phi$ such that $\nabla\varphi = \nabla^\perp\Phi$, by

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega_\rho \\ \dfrac{\partial \Phi}{\partial n} & \text{given on } \partial\Omega \text{ or } \phi = 0 \text{ on } \partial\Omega \text{ for Neumann b.c.} \\ \displaystyle\int_{\partial B(a_i,\rho)} \frac{\partial \Phi}{\partial n} = 2\pi d_i. \end{cases}$$

As $\rho \to 0$, $\Phi$ behaves like the solution of

$$\begin{cases} \Delta\Phi_0 = 2\pi \sum_i d_i \delta_{a_i} \quad \Omega \\ \dfrac{\partial \Phi_0}{\partial n} \text{ given} \quad \text{or} \quad \Phi_0 = 0 \text{ on } \partial\Omega \text{ for Neumann b.c.} \end{cases}$$

Introducing the Green's kernel associated to $\Omega$ (with the right boundary condition), which has a $\log|x-y|$ type singularity, and its regular part $S(x,y) = 2\pi G(x,y) - \log|x-y|$, we have

$$\Phi_0(x) = 2\pi \sum_j d_j G(x, a_j).$$

With this relation, $\int_\Omega |\nabla\Phi_0|^2$ is infinite but would write formally like

$$\int_\Omega |\nabla\Phi_0|^2 = -2\pi \sum_i d_i \Phi_0(a_i) = -4\pi^2 \sum_{i,j} d_i d_j G(a_i, a_j).$$

Now we wish to estimate $\frac{1}{2}\int_{\Omega_\rho} |\nabla\varphi|^2 = \frac{1}{2}\int_{\Omega_\rho} |\nabla\Phi|^2$ and it is approximately equal to

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla\Phi_0|^2 \simeq \pi \sum_i d_i^2 \log\frac{1}{\rho} + W_{\mathbf{d}}(a_1, \cdots, a_n) + o(1) \quad \text{as } \rho \to 0 \tag{9}$$

where

$$W_{\mathbf{d}}(a_1, \cdots, a_n) = -\pi \sum_{i \neq j} d_i d_j \log|a_i - a_j| - \pi \sum_{i,j} S(a_i, a_j). \tag{10}$$

The function $W$ was called the *renormalized energy* in [2]. It contains the (logarithmic) interaction energy between the vortices: we see that vortices with degrees of same sign repel each other while vortices with degrees of opposite signs attract. The $d^2 \log\frac{1}{\rho}$ term corresponds to the self-interaction, or cost of the vortex of core of size $\rho$, it is what replaces the infinite term in the formal calculation.

Now (9) is a good estimate for the optimal energy outside of the holes, while the energy in holes of size $\rho$ was estimated through (5). Combining these estimates, we are led to the major result of [2]:

**Theorem 3 ([2])** *Assume $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$ and $u_\varepsilon = g$ on $\partial\Omega$, with deg $g \neq 0$. Then, up to extraction,*

$$\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle \rightharpoonup 2\pi \sum_{i=1}^{n} d_i \delta_{a_i} \quad d_i \in \mathbb{Z}$$

*and*

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_{i=1}^{n} |d_i| \log \frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1, \ldots, a_n) + o(1) \quad \textit{as} \quad \varepsilon \to 0$$

So for a given degree $d > 0$ on the boundary, in order to minimize the energy, one needs to choose $d$ vortices of degree $+1$, and then to minimize the remaining interaction term $W$ which is independent of $\varepsilon$ and governs the locations of the limiting vortices.

From now on, we will reduce to the case $d_i = \pm 1$ and will use

**Theorem 4** *1. Assume $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$ and $u_\varepsilon = g$ on $\partial\Omega$ or $\frac{\partial u_\varepsilon}{\partial n} = 0$ on $\partial\Omega$, then, up to extraction,*

$$\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle \rightharpoonup 2\pi \sum_{i=1}^{n} d_i \delta_{a_i}$$

*and if $\forall i$, $d_i = \pm 1$,*

$$\varliminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - \pi n |\log \varepsilon| \geq W_{\mathbf{d}}(a_1, \cdots, a_n).$$

*2. For all $(a_i, d_i)$, $d_i = \pm 1$, there exists $u_\varepsilon$ such that*

$$\varlimsup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - \pi n |\log \varepsilon| \leq W_{\mathbf{d}}(a_1, \cdots, a_n).$$

Phrased this way, it is a result of $\Gamma$-convergence of $E_\varepsilon - \pi n|\log \varepsilon|$, and the $\Gamma$-limit, $W_{\mathbf{d}}$ is *nontrivial*. We thus reduce minimizing $E_\varepsilon$ to minimizing $W_{\mathbf{d}}$ which is a *finite-dimensional* problem (interaction of point charges). Thus we see why it is interesting to study this asymptotic limit $\varepsilon \to 0$ because the vortices become point-like and the problem reduces to a finite-dimensional one.

## 2 The abstract result for $\Gamma$-convergence of gradient-flows

### 2.1 The abstract situation

Let $E_\varepsilon$ be again a family of functionals defined on $\mathcal{M}_\varepsilon$ (see [19] for an idea of what kind of space $\mathcal{M}_\varepsilon$ should be...) and $F$ be a functional defined on $\mathcal{N}$ such that $E_\varepsilon$ $\Gamma$-converges to $F$ for the sense of convergence $S$ (in the sense of Definition 1). If we consider a solution of the gradient-flow (or steepest descent) of $E_\varepsilon$ on $\mathcal{M}_\varepsilon$ i.e.

$$\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon),$$

does $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ for some $u(t)$ and more importantly, does $u(t)$ satisfy $\partial_t u = -\nabla F(u)$? An example with a positive answer is that of the functional $M_\varepsilon$ whose $L^2$ gradient-flow is the Allen-Cahn equation

$$\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \tag{11}$$

We saw that $M_\varepsilon \xrightarrow{\Gamma} F = \frac{8}{3}\text{per }(\gamma)$. In fact it is true that solutions of the Allen-Cahn equation converge to interfaces which evolve according to the gradient flow of that perimeter functional $F$, which is *mean-curvature flow*. This result is a delicate one, which has been proved with PDE methods (see [8, 6, 9, 10]).

Let us point out that the answer is in general *negative* without further assumptions. Indeed, a necessary condition is that critical points of $E_\varepsilon$ should converge to critical points of $F$, but we already mentioned that when $E_\varepsilon$ $\Gamma$-converges to $F$, this is not necessarily true.

We are searching for
- an abstract result
- an energy-based method
- new extra conditions for convergence to occur.

Observe that these have to involve the $C^1$ (or tangent) structure of the energy landscape, i.e. be conditions on the derivatives of the energy and not only of the energies themselves (otherwise it is easy to perturb the energy by a small perturbation in $C^0$ which adds new critical points which do not converge to critical points).

We have been sloppy until now, by writing $\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$ and calling this the gradient-flow of $E_\varepsilon$. Since we are in infinite dimensions (in general), we need to specify what we mean by gradient, i.e. gradient with respect to which structure. There are many possible choices, each leading to a different gradient-flow. For example, the Allen-Cahn equation (11) above is the gradient flow for $M_\varepsilon$ for the $L^2$ structure. We could consider other structures, such as the gradient-flow with respect to the $H^{-1}$ structure, it is then a totally different dynamics, called the Cahn-Hilliard equation. So, when looking for a result of convergence, we need to specify what the structure for the limiting flow should be (recall that the limiting flow is not taken in the same space, $u_\varepsilon \in \mathcal{M}_\varepsilon \stackrel{S}{\rightharpoonup} u \in \mathcal{N} \neq \mathcal{M}_\varepsilon$.) Another element that should come into play is possible time-rescalings as we pass to the limit $\varepsilon \to 0$.

## 2.2 The result

For simplicity we will reduce to the following case: $E_\varepsilon$ is family of $C^1$ functionals defined over $\mathcal{M}$, an open subset of a Banach space $\mathcal{B}$ continuously embedded into a Hilbert space $X_\varepsilon$ (or of an affine space associated to a Banach). We assume $E_\varepsilon \xrightarrow{\Gamma} F$, with $F$ a $C^1$ functional defined over $\mathcal{N}$, open set of a finite-dimensional vector space $\mathcal{B}'$ embedded into a finite-dimensional Hilbert space $Y$.

**Definition 18** *$E_\varepsilon$ $\Gamma$-converges along the trajectory $u_\varepsilon(t)$ ($t \in [0,T)$) in the sense $S$ to $F$ if there exists $u(t) \in \mathcal{N}$ and a subsequence (still denoted $u_\varepsilon$) such that $\forall t \in [0,T)$, $u_\varepsilon(t) \xrightarrow{S} u(t)$ and*

$$\forall t \in [0,T) \quad \varliminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon(t)) \geq F(u(t)).$$

**Definition 19** *If $dE_\varepsilon(u)$, differential of $E_\varepsilon$ at $u$, is linear continuous on $X_\varepsilon$, it is uniquely represented by a vector in $X_\varepsilon$, denote it by $\nabla_{X_\varepsilon} E_\varepsilon(u)$ (gradient for the structure $X_\varepsilon$), characterized by*

$$\forall \phi \in X_\varepsilon \quad \frac{d}{dt}_{|t=0} E_\varepsilon(u + t\phi) = dE_\varepsilon(u) \cdot \phi = \langle \nabla_{X_\varepsilon} E_\varepsilon(u), \phi \rangle_{X_\varepsilon}.$$

*If this gradient does not exist, we use the convention $\|\nabla_{X_\varepsilon} E_\varepsilon(u)\|_{X_\varepsilon} = +\infty$*

For example, for the dynamics of Ginzburg-Landau, we wish to study the equation

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2). \tag{12}$$

Let us see how to fit into the previous framework. $E_\varepsilon$ is defined on $H^1(\Omega, \mathbb{C})$, we take $\mathcal{B} = H^1(\Omega, \mathbb{C}) \subset L^2(\Omega)$ and we define the $X_\varepsilon$ structure by

$$\| \cdot \|^2_{X_\varepsilon} = \frac{1}{|\log \varepsilon|} \int_\Omega | \cdot |^2 = \frac{1}{|\log \varepsilon|} \| \cdot \|^2_{L^2(\Omega)},$$

i.e. a rescaled version of $L^2$. Then $\mathcal{B}$ embeds continuously into $X_\varepsilon$, and the gradient for the structure $X_\varepsilon$ is

$$\nabla_{X_\varepsilon} E_\varepsilon(u) = -|\log \varepsilon| \left( \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \right).$$

Indeed

$$\begin{aligned} dE_\varepsilon(u) \cdot \phi &= \int_\Omega \phi \left( -\Delta u - \frac{u}{\varepsilon^2}(1 - |u|^2) \right) \\ &= \frac{1}{|\log \varepsilon|} \int_\Omega \phi \left( -|\log \varepsilon| \left( \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \right) \right) = \langle \phi, \nabla_{X_\varepsilon} E_\varepsilon(u) \rangle_{X_\varepsilon}. \end{aligned}$$

Then the PDE (12) is indeed exactly $\partial_t u = -\nabla_{X_\varepsilon} E_\varepsilon(u)$ i.e. the gradient-flow for the structure $X_\varepsilon$.

Recall that if a solution of

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \tag{13}$$

is smooth enough, we have

$$\begin{aligned}\langle \partial_t u_\varepsilon, \partial_t u_\varepsilon \rangle_{X_\varepsilon} &= -\langle \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \partial_t u_\varepsilon \rangle_{X_\varepsilon} \\ &= -\partial_t E_\varepsilon(u_\varepsilon(t)) \\ \int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 \, dt &= E_\varepsilon\left(u_\varepsilon(0)\right) - E_\varepsilon(u_\varepsilon(T)).\end{aligned}$$

**Definition 20** *A solution of the gradient-flow for $E_\varepsilon$ with respect to the structure $X_\varepsilon$ on $[0,T)$ is a map $u_\varepsilon \in H^1([0,T), X_\varepsilon)$ such that*

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \in X_\varepsilon \quad \textit{for a.e. } t \in [0,T).$$

*Such a solution is conservative if $\forall t \in [0,T)$*

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2 \, ds$$

*(this is true if $u_\varepsilon$ is smooth enough). If $u_\varepsilon$ is such a family of solutions on $[0,T)$ and $E_\varepsilon$ $\Gamma$-converges to $F$ along $u_\varepsilon(t)$ (in the sense of Definition 2), we define the energy-excess $D(t)$ by $D_\varepsilon(t) = E_\varepsilon(u_\varepsilon(t)) - F(u(t)) \geq o(1)$ and*

$$D(t) = \overline{\lim_{\varepsilon \to 0}} D_\varepsilon(t) \geq 0.$$

*A family of solutions of the gradient flow is said to be well-prepared initially if $D(0) = 0$.*

Recall that $F$ is always a lower bound for $E_\varepsilon$. Also, it is always possible to have well-prepared initial data from assertion 2) (the construction part) of the $\Gamma$-convergence definition, Definition 1.

We define similarly the gradient-flow for $F$ for the structure $Y$. We can now state the abstract result.

**Theorem 5 ([19])** *Let $E_\varepsilon$ and $F$ be $C^1$ functionals over $\mathcal{M}$ and $\mathcal{N}$ respectively, $E_\varepsilon \xrightarrow{\Gamma} F$, and let $u_\varepsilon$ be a family of conservative solutions of the flow of $E_\varepsilon$*

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \quad \textit{on} \quad [0,T) \tag{14}$$

*with $u_\varepsilon(0) \overset{S}{\rightharpoonup} u_0$, along which $E_\varepsilon$ $\Gamma$-converges to $F$ in the sense of Definition 2.*

*Assume moreover that 1) and either 2) or 2') below are satisfied:*

1) *(lower bound) For a subsequence such that $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$, we have $u \in H^1((0,T), Y)$ and $\forall s \in [0,T)$*

$$\underline{\lim_{\varepsilon \to 0}} \int_0^s \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2(t) \, dt \geq \int_0^s \|\partial_t u\|_Y^2(t) \, dt. \tag{15}$$

2) *For any $t \in [0,T)$*

$$\underline{\lim_{\varepsilon \to 0}} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(t)\|_{X_\varepsilon}^2 \geq \|\nabla_Y F(u(t))\|_Y^2. \tag{16}$$

*2′) (construction) If* $u_\varepsilon \stackrel{S}{\rightharpoonup} u$*, for any* $V \in Y$*, any* $v$ *defined in a neighborhood of* $0$ *satisfying*

$$\begin{cases} v(0) &= u \\ \partial_t v(0) = V \end{cases}$$

*there exists* $v_\varepsilon(t)$ *such that* $v_\varepsilon(0) = u_\varepsilon$

$$\overline{\lim_{\varepsilon\to 0}} \|\partial_t v_\varepsilon(0)\|^2_{X_\varepsilon} \le \|\partial_t v(0)\|^2_Y = \|V\|^2_Y$$

$$\underline{\lim}_{\varepsilon\to 0} -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \ge -\frac{d}{dt}_{|t=0} F(v) = -\langle \nabla_Y F(u), V\rangle_Y$$

*Then if* $D(0) = 0$ *(i.e. the solution is well-prepared initially) we have* $D(t) = 0\ \forall t \in [0,T)$*, all inequalities above are equalities and* $\forall t \in [0,T)$*,* $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ *where*

$$\begin{cases} \partial_t u = -\nabla_Y F(u) \\ u(0) = u_0 \end{cases}$$

*i.e.* $u$ *is a solution of the gradient flow for* $F$ *for the structure* $Y$*.*

### 2.3 Interpretation

This theorem means that under conditions 1) and 2), or 1) and 2’) (since 2’) implies 2)), solutions of the gradient flow of $E_\varepsilon$ for the structure $X_\varepsilon$ converge to solutions of limiting gradient-flow (for the structure $Y$) if well-prepared. Let us make a few additional comments:

1. The limiting structure $Y$ is somehow embedded in the conditions 1) and 2). The time rescalings are embedded in $X_\varepsilon$.
2. In general we expect 1) and 2) to be satisfied for any $u_\varepsilon \stackrel{S}{\rightharpoonup} u$ or $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} u(t)$ not necessarily solutions (here we required it only for solutions)
3. 1) and 2) do provide the extra $C^1$ order conditions on $\Gamma$-convergence. 2) in particular implies that critical points converge to critical points.
4. The difficulty is not in proving this theorem but in proving that in specific cases the conditions hold.

### 2.4 Idea of the proof

Let us see how 1) and 2) imply the result. We assume

$$\begin{cases} \partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \\ \underline{\lim}_{\varepsilon\to 0} \displaystyle\int_0^s \|\partial_t u_\varepsilon\|^2_{X_\varepsilon}\, dt \ge \int_0^s \|\partial_t u\|^2_Y \\ \underline{\lim}_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} \ge \|\nabla_Y F(u)\|^2_Y \\ u_\varepsilon(0) \stackrel{S}{\rightharpoonup} u_0, \quad \lim E_\varepsilon(u_\varepsilon(0)) = F(u_0) \end{cases}$$

Then, for all $t < T$ we may write

$$\begin{aligned}
E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) &= -\int_0^t \langle \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(s)), \partial_t u_\varepsilon(s)\rangle_{X_\varepsilon}\, ds \\
&= \frac{1}{2}\int_0^t \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 + \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2\, ds \\
&\geq \frac{1}{2}\int_0^t \|\nabla_Y F(u)\|_Y^2 |\|\partial_t u\|_Y^2\, ds - o(1) \\
&\geq \int_0^t -\langle \nabla_Y F(u(s)), \partial_t u(s)\rangle_Y\, ds - o(1) \qquad (17) \\
&= F(u(0)) - F(u(t)) - o(1)
\end{aligned}$$

hence

$$F(u(0)) - F(u(t)) \leq E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) + o(1)$$

But by well-preparedness $E_\varepsilon(u_\varepsilon(0)) = F(u(0)) + o(1)$ thus

$$E_\varepsilon(u_\varepsilon(t)) \leq F(u(t)) + o(1).$$

But, $E_\varepsilon \xrightarrow{\Gamma} F$ implies $\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) \geq F(u(t))$ therefore we must have equality everywhere and in particular equality in (17), that is

$$\frac{1}{2}\int_0^t \|\nabla_Y F(u)\|_Y^2 + \|\partial_t u\|_Y^2\, ds = \int_0^t \langle -\nabla_Y F(u(s)), \partial_t u(s)\rangle_Y$$

or

$$\int_0^t \|\nabla_Y F(u) + \partial_t u\|_Y^2\, ds = 0.$$

Hence, we conclude that $\partial_t u = -\nabla_Y F(u)$, $\forall t$.

The idea is thus to show that the energy decreases at least of the amount expected (i.e. the amount of decrease of $F$), on the other hand it cannot decrease more because of the $\Gamma$-convergence, hence it decreases exactly of the amount expected, all along the trajectory.

*Proof of 2') $\Longrightarrow$ 2)* (2') is a constructive proof of 2)). Observe that here $u_\varepsilon$ does not depend on time.

For every $V \in Y$ we may pick $v(t)$ such that

$$\begin{cases} v(0) &= u \\ \partial_t v(0) &= V \end{cases}$$

i.e. pick a tangent curve to $V$ at $u$. We assume there exists (we can construct) $v_\varepsilon(t)$ such that

$$\begin{cases} v_\varepsilon(0) = u_\varepsilon \\ \overline{\lim}_{\varepsilon\to 0} \|\partial_t v_\varepsilon(0)\|_{X_\varepsilon}^2 \leq \|V\|_Y^2 \\ \underline{\lim}_{\varepsilon\to 0} -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \geq -\frac{d}{dt}_{|t=0} F(v) = -\langle \nabla_Y F(u), V\rangle_Y \end{cases}$$

that is a curve $v_\varepsilon(t)$ along which the energy decreases of at least the desired amount. Then, choosing $V = -\nabla_Y F(u)$, we have

$$\langle -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \partial_t v_\varepsilon \rangle_{X_\varepsilon} = -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \geq -\langle \nabla_Y F(u), V\rangle_Y = \|\nabla_Y F(u)\|_Y^2$$

thus

$$\begin{aligned} \|\nabla_Y F(u)\|_Y^2 &\leq \langle -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon), \partial_t v_\varepsilon(0))\rangle_{X_\varepsilon} \\ &\leq \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \|\partial_t v_\varepsilon(0)\|_{X_\varepsilon} \\ &\leq \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} (\|V\|_Y + o(1)) \end{aligned}$$

Recalling that $V = -\nabla_Y F(u)$, we conclude that

$$\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \geq \|\nabla_Y F(u)\|_Y + o(1).$$

The idea was to rely on the fact that steepest descent is characterized as the evolution which maximizes the energy-decrease for a given $\|\partial_t u_\varepsilon\|^2$. We compare it to a test-evolution obtained by "pushing" $u_\varepsilon$ in the direction $V$ (and in fact choose the steepest descent direction $V = -\nabla F(u)$), i.e. find a curve $v(t)$ and "lift it" to a curve $v_\varepsilon$ that pushes $u_\varepsilon$ in direction $-\nabla_Y F(u)$ with a decrease of energy of at least the expected one, and a cost $\|\partial_t v_\varepsilon\|^2$ which is at most the expected one. We can in fact achieve this in such a way that $\partial_t u_\varepsilon(0)$ depends linearly on $V$. In "pedantic" terms, we show that there exists a linear embedding

$$\begin{aligned} \mathcal{I}_\varepsilon : T_u\mathcal{N} &\longrightarrow T_{u_\varepsilon}\mathcal{M} \\ V &\mapsto \partial_t v_\varepsilon(0) \end{aligned}$$

which is an "almost-isometry" in the sense:

$$\lim_{\varepsilon\to 0} \|\mathcal{I}_\varepsilon(V)\|_{X_\varepsilon} = \|V\|_Y \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon^* \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) = \nabla_Y F(u).$$

### 2.5 Application to Ginzburg-Landau

In order to retrieve the dynamical law for vortices, we need to prove that conditions 1) and 2') of Theorem 5 can be proved for Ginzburg-Landau. As seen in Theorem 4, we need to consider the energies

$$F_\varepsilon(u) = E_\varepsilon(u) - \pi n|\log\,\varepsilon| = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 - \pi n|\log\varepsilon| \quad (18)$$

and $F = W$ (the renormalized energy) so that $F_\varepsilon \xrightarrow{\Gamma} F$. The structures we need are

$$\|\cdot\|_{X_\varepsilon}^2 = \frac{1}{|\log\varepsilon|}\|\cdot\|_{L^2(\Omega)}^2$$

$$\mathcal{N} = \Omega^n \backslash \text{diagonals} \quad (19)$$

$$\|\cdot\|_Y^2 = \frac{1}{\pi}\|\cdot\|_{(\mathbb{R}^2)^n}^2 \quad (20)$$

and a prescribed number of vortices of a priori fixed degrees $\pm 1$. Applying Theorem 5, we retrieve the dynamical law (first established by Lin and Jerrard-Soner) that the vortices flow according to a rescaled gradient-flow of the renormalized energy:

**Theorem 6 ([15, 12, 19])** *Let $u_\varepsilon$ be a family of solutions of*

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)$$

*with either*

$$\left| \begin{array}{ll} u_\varepsilon = g & \text{on } \partial\Omega \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \end{array} \right.$$

*such that*

$$\operatorname{curl} \langle i u_\varepsilon, \nabla u_\varepsilon \rangle (0) \rightharpoonup 2\pi \sum_{i=1}^{n} d_i \delta_{a_i^0} \qquad \text{as } \varepsilon \to 0$$

*with $a_i^0$ distinct points in $\Omega$, $d_i = \pm 1$, and*

$$E_\varepsilon(u_\varepsilon)(0) - \pi n |\log \varepsilon| \leq W_{\mathbf{d}}(a_i^0) + o(1). \tag{21}$$

*Then there exists $T^* > 0$ such that $\forall t \in [0, T^*)$,*

$$\operatorname{curl} \langle i u, \nabla u \rangle (t) \rightharpoonup 2\pi \sum_{i=1}^{n} d_i \delta_{a_i(t)}$$

*as $\varepsilon \to 0$, with*

$$\begin{cases} \dfrac{da_i}{dt} = -\dfrac{1}{\pi} \nabla_i W_{\mathbf{d}}(a_1(t), \cdots, a_n(t)) \\ a_i(0) = a_i^0 \end{cases} \tag{22}$$

*where $T^*$ is the minimum of the collision time and exit time (from $\Omega$) of the vortices under this law. Moreover $D(t) = 0$ for every $t < T^*$.*

Thus, as expected, vortices move along the gradient flow for their interaction $W$, and this reduces the PDE to a finite dimensional evolution (a system of ODE's). Thus result was obtained in [15, 12] (also in [23] for a certain regime with magnetic field), but with PDE methods, it is reproven in [19] with the $\Gamma$-convergence energetic method exposed here.

### 2.6 Remarks

1. The result holds as long as the number of vortices remains the initial one (so that the limiting configuration $u = (a_1, \cdots, a_n)$ belongs to the same space $\mathcal{N}$). It ceases to apply when there are vortex-collisions or some vortex exits the domain under the law (22), even though these can happen. Then a further analysis is required, see Section 4.1 below.

2. Under the same hypotheses, if $u_\varepsilon$ is a solution of the time-rescaled gradient flow $\partial_t u_\varepsilon = -\lambda_\varepsilon \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)$ with $D(0) = 0$ then if $\lambda_\varepsilon \ll 1$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u_0, \forall t$ i.e. there is no motion; while if $\lambda_\varepsilon \gg 1$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u, \forall t$ with $\nabla_Y F(u) = 0$ i.e. there is instantaneous motion to a critical point. Thus, we see that the structure $X_\varepsilon$ and the relation 1) in Theorem 5 contain the right time-rescaling to see finite-time motion in the limit. For Ginzburg-Landau without magnetic field, it is necessary to accelerate the time by a $|\log \varepsilon|$ factor in order to see motion of the vortices (this is due to the fact that the renormalized energy $W$ which drives the motion is a lower order term in the energy).
3. We can weaken conditions 1) and 2) to

$$\varliminf_{\varepsilon\to 0} \int_0^t \|\partial_t u_\varepsilon\|^2 \geq \int_0^t \|\partial_t u\|^2 - O(D(t))$$
$$\varliminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} \geq \|\nabla F(u)\|^2_Y - O(D(t))$$

   where $D(t)$ is the energy-excess, and handle the terms in $D(t)$ in the proof via a Gronwall's lemma (finally obtaining that $D(t) \equiv 0$ if $D(0) = 0$).
4. The method should and can be extended to infinite-dimensional limiting spaces and to the case where the Hilbert structures $X_\varepsilon$ and $Y$ (in particular $Y$) depend on the point, such as $Y_u = L^2_u$, forming a sort of Hilbert manifold structure. It is thus interesting to see how, through $\Gamma$-convergence, the structures underlying the gradient-flows can become "curved" at the limit, even though they are not curved originally at the $\varepsilon$ level, and also become possibly nonsmooth and nondifferentiable. In fact we can write down an analogue abstract result using the theory of "minimizing movements" of De Giorgi formalized by Ambrosio-Gigli-Savarè [1], a notion of gradient flows on structures which are not differentiable but simply metric structures.
5. The method works for Ginzburg-Landau with or without magnetic field as long as the number of vortices remains bounded. It is more difficult to apply it to other models such as Allen-Cahn, or 3D Ginzburg-Landau, because what misses is a more precise result and understanding on the profile of the defect during the dynamics. For example, for Allen-Cahn, we need to know that the energy-density remains proportional to the length of the underlying limiting curve during the dynamics (which is true a posteriori). It is also an open problem to apply it when the number of vortices is unbounded as $\varepsilon \to 0$.

# 3 Proof of the additional conditions for Ginzburg-Landau

## 3.1 A product-estimate for Ginzburg-Landau

The relation 1) which relates the velocity of underlying vortices to $\partial_t u_\varepsilon$ can be read

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \int_{[0,t]\times\Omega} |\partial_t u_\varepsilon|^2 \, ds \geq \pi \sum_i \int_0^t |d_t \, a_i|^2 \, ds \tag{23}$$

assuming $\operatorname{curl} \langle iu_\varepsilon, \nabla u_\varepsilon\rangle(t) \rightharpoonup 2\pi \sum_i d_i \delta_{a_i(t)}$, as $\varepsilon \to 0$, $\forall t$. This turns out to hold as a general relation, without asking the configurations to solve any particular equation. It is related to the topological nature of the vortices.

It can be embedded into the more general class of results of lower-bounds for Ginzburg-Landau functionals. The setting is now $\Omega$ a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ (we will need $n = 3$) and still $E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2$. We define the "current" $ju$ associated to $u$ as the 1-form $ju = \langle iu, du\rangle = \sum_k \langle iu, \partial_k \, u\rangle dx_k$. Then the Jacobian $Ju$ is the 2-form

$$Ju = \frac{1}{2} d(ju) = \frac{1}{2} d\langle iu, du\rangle$$

It can be identified to an $(n-2)$-dimensional current through

$$Ju(\phi) = \frac{1}{2}\int Ju \wedge \phi \, dx$$

for $\phi$ an $(n-2)$-form. This current corresponds to the vorticity lines in 3D. For example if $u = e^{i\phi}$ with singular set $\gamma$ straight line parallel to the $z$ axis and a degree $D$ around $\gamma$, then $Ju = \pi D(dx \wedge dy) \lfloor \gamma$ i.e. given test vector-fields $X$ and $Y$,

$$Ju(X,Y) = D \int_\gamma \pi \left(\frac{X_1 Y_2 - X_2 Y_1}{2}\right)$$

The total variation of $Ju$ is then $|Ju| = \pi |D| \mathcal{H}^1(\gamma)$, multiple of the length of the line.

**Theorem 7 ([18])** *Let $u_\varepsilon$ be a family of $H^1(\Omega,\mathbb{C})$ such that*

$$E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$$

*then up to extraction, $\forall \beta > 0$, $Ju_\varepsilon \rightharpoonup J$ in $(C_C^{0,\beta}(\Omega))^*$, with $\frac{J}{\pi}$ an $(n-2)$-dimensional rectifiable integer-multiplicity current (see [13]) and for every $X, Y$ continuous compactly supported vector fields, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \sqrt{\int_\Omega |X \cdot \nabla u_\varepsilon|^2 \int_\Omega |Y \cdot \nabla u_\varepsilon|^2} \geq \left| \int_\Omega J(X,Y) \right|. \tag{24}$$

As a first corollary, in 2D, taking $X = e_1, Y = e_2$ an orthonormal basis, we find

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \sqrt{\int_\Omega |\partial_1 u_\varepsilon|^2 \int_\Omega |\partial_2 u_\varepsilon|^2} \geq \pi \sum_i |d_i|$$

This estimate implies the estimate $\varliminf_{\varepsilon\to 0} \frac{1}{|\log\ \varepsilon|} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} \geq \pi \sum_i |d_i|$ but is sharper. It implies in particular that if $\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \leq \pi \sum_i |d_i||\log\ \varepsilon|$ and $d_i = \pm 1$, then

$$\forall X, \quad \lim_{\varepsilon\to 0} \frac{1}{|\log\ \varepsilon|} \int_\Omega |\nabla u_\varepsilon \cdot X|^2 = \pi \sum_i |X(a_i)|^2 \tag{25}$$

i.e. there is isotropy in the repartition of the energy along different directions.

In 3D, taking $X \perp Y$ and maximizing the right-hand side of (24) over $|X|, |Y| \leq 1$ we find

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log\ \varepsilon|} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} \geq |J|(\Omega),$$

an estimate that was previously proved in [13].

*Remark:* Our result extends to higher energies $E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon|$ with $N_\varepsilon$ unbounded, in that case we just need to rescale by $N_\varepsilon$ and replace $J$ by the limit of $\frac{Ju_\varepsilon}{N_\varepsilon}$.

### 3.2 Idea of the proof

The method consists in reducing to two dimensions. By using partitions of unity, we can assume that $X$ and $Y$ are locally constant. We may then work in an open set $U$ where $X$ and $Y$ are constant. If they are not parallel, they define a planar direction (if they are then $J(X,Y) = 0$ and there is nothing to prove). We then slice $U$ into planes parallel to that plane. Assume $X = e_1$ and $Y = e_2$ orthonormal vectors. In each plane we have the known 2D lower bounds of the type

$$\frac{1}{2} \int_{\text{plane}\cap U} |\nabla u_\varepsilon|^2 \geq \pi \sum |d_i||\log\ \varepsilon|$$

where $d_i$ is the degree of the boundary of the balls, constructed with the ball-construction method (see [17, 11, 20]). This is possible as long as there is a good bound on the energy on that planar slice, and the number of balls can be unbounded.

The main trick is to observe that this is true for any metric in the plane, and use the metric $\lambda dx + \frac{1}{\lambda} dy$, leading to

$$\frac{1}{2\lambda} \int_{plane\cap U} |\partial_1 u_\varepsilon|^2 + \frac{\lambda}{2} \int_{plane\cap U} |\partial_2 u_\varepsilon|^2 \geq \pi \sum |d_i|| \log \varepsilon|$$

Integrating with respect to the slices yields

$$\frac{1}{2\lambda}\int_U |\nabla u_\varepsilon \cdot X|^2 + \frac{\lambda}{2}\int_U |\nabla u_\varepsilon \cdot Y|^2 \geq \left|\int_U J(X,Y)\right| |\log \varepsilon|.$$

Optimizing with respect to $\lambda$, we conclude that

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|}\int_U |\nabla u_\varepsilon \cdot X|^2 \int_U |\nabla u_\varepsilon \cdot Y|^2 \geq \left|\int_U J(X,Y)\right|$$

and we may finish by adding these estimates thanks to the partitions of unity.

### 3.3 Application to the dynamics

In order to deduce a result for the dynamics in 2D, the idea is to use this theorem in dimension $n = 3$ with 2 coordinates corresponding to space coordinates and 1 coordinate corresponding to the time coordinate (this can be done in any dimension, but we restrict to 2D here for the sake of simplicity). The vortex-lines in 3D are then the trajectories in time of the vortex-points in 2D, and clearly the length of these lines is somehow related to the velocity of these points. Splitting the coordinates, we write

$$\begin{aligned} ju &= \langle iu, \partial_t u\rangle\, dt + \langle iu, d_{space}\, u\rangle \\ Ju_\varepsilon &= \underbrace{\sum_{i=1}^2 V_i\, dt \wedge dx_i}_{V_\varepsilon} + \underbrace{\frac{1}{2} d_{space}\langle iu, d_{space} u\rangle}_{\mu_\varepsilon \to \pi \sum d_i \delta_{a_i(t)}} \end{aligned}$$

**Theorem 8 ([18])** *Let $u_\varepsilon(t,x)$ be defined over $[0,T]\times\Omega$ ($\Omega \subset \mathbb{R}^n$, here $n = 2$) and such that*

$$\begin{cases} \forall t\ E_\varepsilon(u_\varepsilon(t)) \leq C|\log \varepsilon| \\ \displaystyle\int_{[0,T]\times\Omega} |\partial_t u_\varepsilon|^2 \leq C|\log \varepsilon| \end{cases}$$

*then*

$$\mu_\varepsilon \rightharpoonup \mu \quad in\ (C_C^{0,\gamma})^*, \quad \mu \in L^\infty([0,T], C_0^0(\Omega)^*), \quad \mu = \pi \sum_i d_i\, \delta_{a_i(t)}$$

$$V_\varepsilon \rightharpoonup V, \qquad V \in L^\infty([0,T], C_0^0(\Omega)^*)$$

*with*

$$d_t\mu + \text{div}\ V = 0.$$

*Moreover, $\forall X \in C_C^0([0,T]\times\Omega, \mathbb{R}^n)$, and $f \in C_C^0([0,T]\times\Omega)$,*

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|}\sqrt{\int_{[0,T]\times\Omega} |X\cdot\nabla u_\varepsilon|^2 \int_{[0,T]\times\Omega} f^2|\partial_t u_\varepsilon|^2} \geq \left|\int_{[0,T]\times\Omega} V\cdot fX\right|$$

In 2D, the vector $V = (V_1, V_2)$ really is $\pi \sum_i d_i (\partial_t a_i)\delta_{a_i(t)}$, such that

$$\partial_t \left( \pi \sum_i d_i \, \delta_{a_i(t)} \right) + \operatorname{div} V = 0$$

**Corollary 1** *If in addition $d_i = \pm 1$ and $\forall t$, $\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \leq \pi \left(\sum_i |d_i|\right) |\log \varepsilon| (1 + o(1))$, then for all intervals $[t_1, t_2)$ on which the $a_i$'s remain distinct, we have*

$$\varliminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{\Omega \times [t_1, t_2]} |\partial_t u_\varepsilon|^2 \geq \pi \sum_i \int_{t_1}^{t_2} |\partial_t a_i|^2 \, dt$$

This is the desired estimate 1) in Theorem 5. To prove this corollary, recall that from (25), if $E_\varepsilon(u_\varepsilon) \simeq \pi n |\log \varepsilon|$ then $\frac{1}{|\log \varepsilon|}\int_\Omega |X \cdot \nabla u_\varepsilon|^2 \simeq \pi \sum_i |X(a_i)|^2$ and optimizing over $X$ and $f$ gives the $L^2$ bound on $V$.

### 3.4 Proof of the construction 2′)

We wish to prove that 2') holds for Ginzburg-Landau so that we deduce 2) i.e. if $u_\varepsilon \overset{S}{\rightharpoonup} u$ then $\varliminf_{\varepsilon \to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \geq \|\nabla W(u)\|_Y$.

Observe that this is a static result. We thus assume that curl $\langle iu_\varepsilon, \nabla u_\varepsilon \rangle \rightharpoonup 2\pi \sum_i d_i \delta_{a_i}$, where $u = ((a_1, d_1), \cdots, (a_n, d_n))$ and may consider disjoint balls $B(a_i, \rho)$ of fixed radius $\rho$. If $\|\nabla_{X_\varepsilon} E(u_\varepsilon)\|_{X_\varepsilon} \to +\infty$ there is nothing to prove. If $\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} = O(1)$ then we can prove that $D_\varepsilon = o(1)$ where $D_\varepsilon = E_\varepsilon(u_\varepsilon) - \pi n |\log \varepsilon| - W(u)$ is the "energy-excess". The proof of this result (see [22, 21]) relies on the fact that $\|\nabla_{X_\varepsilon} E_\varepsilon(u)\|_{X_\varepsilon} \leq C$ means $\int_\Omega \left| \Delta u + \frac{u}{\varepsilon^2}\left(1 - |u|^2\right) \right|^2 \leq \frac{C}{|\log \varepsilon|} = o(1)$ and one can take advantage of the fact that $u_\varepsilon$ is thus an "almost-solution".

Once this is proved, we may deduce

$$\frac{1}{2}\int_{B(a_i,\rho)} |\nabla u_\varepsilon|^2 = \pi |\log \varepsilon| + O(1)$$

$$\frac{1}{2}\int_{\Omega \setminus \cup_i B(a_i,\rho)} |\nabla |u_\varepsilon||^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \leq D_\varepsilon = o(1) \tag{26}$$

$$\frac{1}{2}\int_{\Omega \setminus \cup_i B(a_i,\rho)} \left| \nabla u_\varepsilon - i u_\varepsilon \nabla^\perp \Phi_0 \right|^2 \leq D_\varepsilon = o(1) \tag{27}$$

where

$$\Delta \Phi_0 = 2\pi \sum_i d_i \delta_{a_i} \quad \text{in } \Omega$$

with the appropriate boundary conditions. The rough idea is that $\nabla \varphi_\varepsilon \simeq \nabla^\perp \Phi_0$ outside of the vortex balls. Through these relations, everything is well-controlled outside the balls and inside the balls we shall only perform a pure translation.

Given $V = (V_1, \ldots, V_n)$, we want to push each $a_i$ in the direction $V_i$. For that purpose, define $\chi_t(x) = x + tV_i$ in each $B_i$, and extend it in a smooth way outside of the $B_i$'s into a family of smooth diffeomorphisms that keep $\partial\Omega$ fixed and are *independent of* $\varepsilon$. Choosing the deformation

$$v_\varepsilon(x,t) = u_\varepsilon(\chi_t^{-1}(x))$$

does the job of pushing the vortices $a_i$ along the direction $V_i$. However it is not enough, and we need to add a phase correction $\psi_t$:

$$v_\varepsilon\left(\chi_t(u), t\right) = u_\varepsilon(x)\, e^{i\psi_t(x)} \tag{28}$$

so that for every $t$, the phase of $v_\varepsilon$ is approximately the optimal one, that is the harmonic conjugate of

$$\Delta\Phi_t = 2\pi \sum_i d_i\, \delta_{a_i(t)} \quad a_i(t) = a_i + tV_i.$$

It is possible to construct $\psi_t$ single-valued, independent of $\varepsilon$, so that

$$\nabla^\perp \Phi_0 + \nabla\psi_t \simeq \nabla^\perp \left(\Phi_t \circ \chi_t\right).$$

We will now check that the $v_\varepsilon$ constructed this way works. First,

$$\begin{aligned}\frac{1}{|\log\,\varepsilon|}\int_\Omega |\partial_t v_\varepsilon|^2(0) &\simeq \frac{1}{|\log\,\varepsilon|}\sum_i \int_{B_i} |V_i\cdot\nabla u_\varepsilon|^2 + o(1)\\ &\simeq \pi\sum_i |V_i|^2 + o(1)\end{aligned}$$

because $\chi_t$ achieves a translation of vector $V_i$ in the $B_i$'s while the contribution outside of the $B_i$'s is negligible; and from the relation (25). The first requirement for 2') is thus fulfilled. Let us check the second requirement, i.e. the energy-decrease rate, by evaluating $\frac{d}{dt}_{|t=0} E_\varepsilon\left(v_\varepsilon(t)\right)$. With a change of variables,

$$\begin{aligned}E_\varepsilon\left(v_\varepsilon(t)\right) &= \frac12\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(1-|v_\varepsilon|^2\right)^2\\ &= \frac12\int_\Omega \left(\left|D\chi_t^{-1}\,\nabla(v_\varepsilon\circ\chi_t)\right|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2\right)|Jac\ \chi_t|.\end{aligned}$$

Now, recall that $\chi_t$ is a translation in $\cup_i B_i$ hence $|Jac\ \chi_t| = cst$ there, while outside of $\cup_i B_i$ there is almost no energy, hence

$$\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}_{|t=0}\int_\Omega \left|D\,\chi_t^{-1}\,\nabla\left(u_\varepsilon\, e^{i\psi_t}\right)\right|^2\,|Jac\ \chi_t| + o(1).$$

Next, we expand $\nabla\left(u_\varepsilon\, e^{i\psi_t}\right)$ as $\nabla u_\varepsilon e^{i\psi_t} + iu_\varepsilon\nabla\psi_t$, expand the squares, and apply $\frac{d}{dt}_{|t=0}$. The crucial fact is that the terms which get differentiated do not depend on $\varepsilon$. For the other terms, we use (27), so that there remains

$$\begin{aligned}\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) &= \int_{\Omega\setminus\cup_i B_i} \left(\frac{d}{dt}_{|t=0} D\chi_t^{-1}\right) \nabla^\perp \Phi_0 \cdot \nabla^\perp \Phi_0 \\ &\quad + \int_{\Omega\setminus\cup_i B_i} \frac{d}{dt}_{|t=0} \nabla\psi_t \cdot \nabla^\perp \Phi_0 \\ &\quad + \frac{1}{2}|\nabla \Phi_0|^2 \frac{d}{dt}_{|t=0} |Jac\ \chi_t| + o(1) \\ &= \frac{d}{dt}_{|t=0} \frac{1}{2} \int_{\Omega\setminus\cup_i B_i} |D\chi_t^{-1}(\nabla^\perp \Phi_0 + \nabla\psi_t)|^2 |Jac\ \chi_t| + o(1)\end{aligned}$$

But observing that $\psi_t$ was constructed in such a way that $\nabla^\perp \Phi_0 + \nabla\psi_t = \nabla^\perp(\Phi_t \circ \chi_t)$, and doing a change of variables again, we find

$$\begin{aligned}\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) &= \frac{d}{dt}_{|t=0} \frac{1}{2} \int_{\Omega\setminus\cup_i B_i} |\nabla \Phi_t|^2 + o(1) \\ &= \frac{d}{dt}_{|t=0} W_{\mathbf{d}}(a_1(t), \cdots, a_n(t)) + o(1)\end{aligned}$$

i.e. the desired result.

## 4 Extensions of the method

### 4.1 Collisions

When there are some positive as well as some negative vortices, the limiting dynamics (22) induces collisions between vortices of opposite signs which attract each other. One expects that those vortices annihilate and the dynamics continues with the remaining ones. This has been treated recently in the papers [4, 5, 21].

One of the things done in [21] was to extend the method presented here to treat the case of collisions. If two vortices of opposite degrees get at a distance $l_\varepsilon = o(1)$ of each other, then it is possible to rescale space in order to have two vortices at distance 1. As long as $|\log l_\varepsilon| \ll |\log \varepsilon|$ the same method we presented above carries through, i.e. we can prove the analogues of conditions 1) and 2), and yields that the dynamics continues with the same type of law (22) but in space-time rescaled coordinates.

When vortices become too close to apply this, we focused on evaluating energy dissipation rates, through the study of the perturbed Ginzburg-Landau equation

$$\Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2) = f_\varepsilon \quad \text{in } \Omega, \tag{29}$$

with Dirichlet or Neumann boundary data, where $f_\varepsilon$ is given in $L^2(\Omega)$ (recall the instantaneous energy-dissipation rate in the dynamics is exactly $|\log \varepsilon| \|f_\varepsilon\|^2_{L^2(\Omega)}$). We prove that the energy-excess (still meaning the difference between $E_\varepsilon - \pi n |\log \varepsilon|$ and the renormalized energy $W$ of the underlying vortices) is essentially controlled by $C\|f_\varepsilon\|_{L^2}^2$. We then show that when $u$ solves (29) and has vortices which become very close, forming what we call an "unbalanced cluster" in the sense that $\sum_i d_i^2 \neq (\sum_i d_i)^2$ in the cluster (see [21] for a precise definition), then the lower bound

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 \geq \min\left(\frac{C}{l^2|\log\varepsilon|}, \frac{C}{l^2\log^2 l}\right) \tag{30}$$

holds. In particular, when vortices get close to each other, say two vortices of opposite degrees for example, then they form an unbalanced cluster of vortices at scale $l =$ their distance, and the relation (30) gives a large energy-dissipation rate (scaling like $1/l^2$). This serves to show that such a situation cannot persist for a long time and we are able to prove that the vortices collide and disappear in time $Cl^2 + o(1)$, with all energy-excess dissipating in that time. Thus after this time $o(1)$, the configuration is again "well-prepared" and Theorem 6 can be applied again, yielding the dynamical law with the remaining vortices, until the next collision, etc...

### 4.2 Second order questions - stability issues

We extended the "$\Gamma$-convergence" method to second order in order to treat stability questions for this 2D Ginzburg-Landau equation. Here is the abstract result, pushing the method of condition 2') to second order. The setting is as in Section II.1. By stable critical point, we mean nonnegative Hessian.

**Theorem 9 ([22])** *Let $u_\varepsilon$ be a family of critical points of $E_\varepsilon$ with $u_\varepsilon \rightharpoonup^S u \in \mathcal{N}$, such that the following holds: for any $V \in \mathcal{B}'$, we can find $v_\varepsilon(t) \in \mathcal{M}$ defined in a neighborhood of $t = 0$, such that $\partial_t v_\varepsilon(0)$ depends on $V$ in a linear and one-to-one manner, and*

$$v_\varepsilon(0) = u_\varepsilon(0) \tag{31}$$

$$\lim_{\varepsilon\to 0} \frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}_{|t=0} F(u + tV) = dF(u).V \tag{32}$$

$$\lim_{\varepsilon\to 0} \frac{d^2}{dt^2}_{|t=0} E_\varepsilon(v_\varepsilon(t)) = \frac{d^2}{dt^2}_{|t=0} F(u + tV) = Q(u)(V). \tag{33}$$

*Then*
*- if (31)-(32) are satisfied, then $u$ is a critical point of $F$*
*- if (31)-(32)-(33) are satisfied, then if $u_\varepsilon$ are stable critical points of $E_\varepsilon$, $u$ is a stable critical point of $F$. More generally, denoting by $n_\varepsilon^+$ the dimension (possibly infinite) of the space spanned by eigenvectors of $D^2E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues, and $n^+$ the dimension of the space spanned by eigenvectors of $D^2F(u)$ associated to positive eigenvalues (resp. $n_\varepsilon^-$ and $n^-$ for negative eigenvalues); for $\varepsilon$ small enough we have*

$$n_\varepsilon^+ \geq n^+ \qquad n_\varepsilon^- \geq n^-.$$

Thus, we reobtain that critical points converge to critical points of the limiting energy $F$ (proved in [2] for Ginzburg-Landau), but in addition we obtain that under certain conditions, stability/instability of the critical point also passes to the limit. The previous result (Theorem 1), was an analysis of the $C^1$ structure of the energy landscape, thus suited to give convergence of gradient-flow and critical points; while this is the $C^2$ analysis of the energy landscape around a critical point.

For the Ginzburg-Landau energy, the construction done in Section 3.4 can be pushed to second order, yielding condition (33). Thus we deduce the corresponding theorem for solutions of (3), (see [22]). An interesting application is for Neumann boundary condition, for which it is known that the corresponding renormalized energy $W$ has *no stable critical point.* Hence from Theorem 9 there can be no stable critical points of $E_\varepsilon$ with vortices (in contrast with the case of $G_\varepsilon$ with nonzero applied magnetic field).

**Theorem 10 ([22])** *Let $u_\varepsilon$ be a family of nonconstant solutions of*

$$\begin{cases} -\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

*(on $\Omega \subset \mathbb{R}^2$ simply connected) such that $E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$; then, for $\varepsilon$ small enough, $u_\varepsilon$ is unstable.*

This is an extension of a result of Jimbo and Sternberg [14] for convex domains.

## References


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# PDE analysis of concentrating energies for the Ginzburg-Landau equation

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These notes are intended as a complement to the lectures given by the author during the summer course *Concentration phenomena in variational problems* which took place in Rome, Università La Sapienza, September 1 - 5 2003. The author wishes to thank Andrea Braides and Valeria Chiadò Piat for their kind invitation and constant help.

The great majority of the material will be taken from [3, 4, 5], but we tried to adopt a more explanatory viewpoint. Some new material is also presented, mainly in Section 4. By no mean we wish to discuss the whole history of the analysis of the Ginzburg-Landau equations here; for this purpose the interested reader is highly encouraged to consult the references in [2, 3, 5].

There are actually many equations which are called the Ginzburg-Landau equations; some have a physical motivation, others don't. Nevertheless, they all have some common mathematical features, and for this reason we will restrict our analysis in these notes to the simplest mathematical case: the static Ginzburg-Landau equation without magnetic field (which is also the less physical!), but in dimension $N \geq 3$. The brief discussion of the time dependent (parabolic and Schrödinger) equations, which was the object of the last lecture, is not covered in these notes.

## 1 Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N,\ N \geq 3$, and $0 < \varepsilon < 1$ be given. A function $u_\varepsilon : \Omega \to \mathbb{C}$ is said to satisfy the Ginzburg-Landau equation $(\mathrm{GL})_\varepsilon$ on $\Omega$ if

$$(\mathrm{GL})_\varepsilon \qquad -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \qquad \text{on } \Omega.$$

We do not prescribe any boundary condition here since we will only discuss results which are local in nature.

To the Ginzburg-Landau equation is associated an energy density

$$e_\varepsilon(u_\varepsilon) = \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2},$$

and a total energy

$$E_\varepsilon(u_\varepsilon) = \int_\Omega e_\varepsilon(u_\varepsilon).$$

In the sequel, we assume that we are given a sequence $\varepsilon \to 0$ and a corresponding sequence of solutions $u_\varepsilon$ of $(\mathrm{GL})_\varepsilon$ on $\Omega$. The main assumption that we will make in some places (see [2] for the motivation) is

$$(H_0) \qquad E_\varepsilon(u_\varepsilon) \le M_0 |\log \varepsilon|,$$

where $M_0$ is some fixed constant not depending on $\varepsilon$.
Under this assumption, the renormalized energy densities

$$\mu_\varepsilon(u_\varepsilon) = \frac{e_\varepsilon(u_\varepsilon)}{|\log \varepsilon|}$$

are uniformly bounded in $L^1(\Omega)$, and we may assume, passing possibly to a subsequence, that

$$\mu_\varepsilon(u_\varepsilon) \to \mu_* \qquad \text{as measures}$$

for some non-negative Radon measure $\mu_*$ on $\Omega$.

**The measure $\mu_*$ is called the energy defect measure or energy concentration measure, and the primary goal of these notes is to present some PDE tools for $(\mathrm{GL})_\varepsilon$ that lead to qualitative and quantitative properties of $\mu_*$.**

In Section 2, after discussing briefly the scaling properties of $(\mathrm{GL})_\varepsilon$, we introduce the monotonicity formula: a simple tool which already allows to obtain important information concerning $\mu_*$. Section 3 is devoted to the clearing-out theorem and to its numerous consequences. The proof of this theorem is too long to be covered in a lecture, and is therefore omitted. Instead, we will concentrate on its consequences concerning $\mu_*$. In section 4, we present some (sometimes new) material concerning pointwise estimates, potential estimates, ..., which will enter into play in Section 5. In this last section, we gather all the information already obtained concerning $\mu_*$, and derive some of its regularity properties as well as a curvature equation.

## 2 The monotonicity formula

### 2.1 Scaling properties

One of the most important properties of $(\mathrm{GL})_\varepsilon$ is its scaling property. Let $0 < R < \frac{1}{\varepsilon}$ and $\epsilon = \varepsilon R$. If $u_\varepsilon$ is a solution of $(\mathrm{GL})_\varepsilon$ on $\Omega$ and $x \in \Omega$, then the rescaled function

$$v_\epsilon(y) = u_\varepsilon(\frac{y-x}{R})$$

is a solution of $(\mathrm{GL})_\epsilon$ on $\Omega_R = R(\Omega - x)$. Notice also that

$$E_\epsilon(v_\epsilon, \Omega_R) = R^{N-2} E_\varepsilon(u_\varepsilon, \Omega). \tag{1}$$

The following definition is motivated by this last equality.

**Definition 21** *Let $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$. The* rescaled energy *of $u_\varepsilon$ in $B(x,r)$ is defined by*

$$\tilde{E}_\varepsilon(u_\varepsilon, x, r) = \frac{1}{r^{N-2}} \int_{B(x,r)} e_\varepsilon(u_\varepsilon).$$

Coming back to (1), we therefore obtain the identity

$$\tilde{E}_\epsilon(v_\epsilon, x, rR) = \tilde{E}_\varepsilon(u_\varepsilon, x, r). \tag{2}$$

## 2.2 The monotonicity formula

We first start with the following identity:

**Lemma 1 (Pohozaev identity)** *If $u_\varepsilon$ is a solution of $(GL)_\varepsilon$ on $B(x,r)$, then*

$$\begin{aligned} \frac{N-2}{2} \int_{B(x,r)} |\nabla u_\varepsilon|^2 + \frac{N}{4\varepsilon^2} \int_{B(x,r)} (1 - |u_\varepsilon|^2)^2 \\ = \int_{\partial B(x,r)} [r \frac{|\nabla_\top u_\varepsilon|^2}{2} - \frac{r}{2} \left| \frac{\partial u_\varepsilon}{\partial n} \right|^2 + \frac{r}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2]. \end{aligned} \tag{3}$$

*Proof.* Integrating by parts, we obtain

$$\begin{aligned} \int_{B(x,r)} -\Delta u_\varepsilon(y)(y-x) \cdot \nabla u_\varepsilon(y)\, dy = \int_{B(x,r)} \nabla u_\varepsilon \cdot \nabla\left((y-x) \cdot \nabla u_\varepsilon\right) \\ - \int_{\partial B(x,r)} r |\partial_r u_\varepsilon|^2. \end{aligned} \tag{4}$$

Notice that

$$\nabla u_\varepsilon \cdot \nabla\left((y-x) \cdot \nabla u_\varepsilon\right) = \frac{1}{2} \nabla\left(|\nabla u_\varepsilon|^2\right) \cdot (y-x) + |\nabla u_\varepsilon|^2.$$

Therefore, integrating by parts once more leads to

$$\begin{aligned} \int_{B(x,r)} -\Delta u_\varepsilon(y)(y-x) \cdot \nabla u_\varepsilon(y)\, dy = -\frac{N-2}{2} \int_{B(x,r)} |\nabla u_\varepsilon|^2 \\ + \frac{1}{2} \int_{\partial B(x,r)} r |\nabla u_\varepsilon|^2 - \int_{\partial B(x,r)} r |\partial_r u_\varepsilon|^2. \end{aligned} \tag{5}$$

On the other hand,

$$\int_{B(x,r)} \frac{1}{\varepsilon^2} u_\varepsilon(1-|u_\varepsilon|^2)(y-x)\cdot\nabla u_\varepsilon = \int_{B(x,r)} -\nabla\left(\frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}\right)\cdot(y-x)$$
$$= N\int_{B(x,r)} \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} - \int_{\partial B(x,r)} r\frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}. \qquad (6)$$

Since $u_\varepsilon$ satisfies $(\mathrm{GL})_\varepsilon$, we obtain the desired result combining (5) and (6).

We are now in position to prove the following proposition, which possesses many consequences, as we will see.

**Proposition 22 (Monotonicity formula)** *Assume $u_\varepsilon$ is a solution of $(GL)_\varepsilon$ in $B(x,R)$, then*

$$\frac{d}{dr}(\tilde{E}_\varepsilon(u_\varepsilon,x,r)) = \frac{1}{r^{N-2}}\int_{\partial B(x,r)}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 + \frac{1}{r^{N-1}}\int_{B(x,r)}\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}$$

*for $0<r<R$.*

*Proof.* First one has,

$$\frac{d}{dr}(E_\varepsilon(u_\varepsilon,x,r)) = \int_{\partial B(x,r)}\frac{|\nabla u_\varepsilon|^2}{2} + \frac{1}{4\varepsilon^2}\int_{\partial B(x,r)}(1-|u_\varepsilon|^2)^2$$
$$= \int_{\partial B(x,r)}\frac{|\nabla_\top u_\varepsilon|^2}{2} + \frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \ .$$

Hence,

$$\frac{d}{dr}(\tilde{E}_\varepsilon(u_\varepsilon,x,r))$$
$$= -\tfrac{N-2}{r^{N-1}}E_\varepsilon(u_\varepsilon,x,r) + \frac{1}{r^{N-2}}\int_{\partial B(x,r)}\frac{|\nabla_\top u_\varepsilon|^2}{2} + \frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}$$
$$= -\left(\tfrac{N-2}{r^{N-1}}\int_{B(x,r)}\frac{|\nabla u_\varepsilon|^2}{2} + \frac{N-2}{4\varepsilon^2 r^{N-1}}\int_{B(x,r)}(1-|u_\varepsilon|^2)^2\right)$$
$$+ \frac{1}{r^{N-2}}\int_{\partial B(x,r)}\frac{|\nabla_\top u_\varepsilon|^2}{2} + \frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2$$
$$= -\left(\frac{1}{r^{N-1}}[\int_{B(x,r)}\frac{N-2}{2}|\nabla u_\varepsilon|^2 + \frac{N}{4\varepsilon^2}\int_{B(x,r)}(1-|u_\varepsilon|^2)^2]\right)$$
$$+ \frac{1}{2\varepsilon^2 r^{N-1}}\int_{B(x,r)}(1-|u_\varepsilon|^2)^2$$
$$+ \frac{1}{r^{N-2}}\int_{\partial B(x,r)}\frac{|\nabla_\top u_\varepsilon|^2}{2} + \frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 .$$

Using Lemma 1, we obtain

$$\begin{aligned}\frac{d}{dr}(\tilde{E}_\varepsilon(u_\varepsilon,x,r)) = &-\left[\frac{1}{r^{N-2}}\int_{\partial B(x,r)}\frac{|\nabla_\top u_\varepsilon|^2}{2}-\frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2+\frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right]\\ &+\frac{1}{r^{N-2}}\left[\int_{\partial B(x,r)}\frac{|\nabla_\top u_\varepsilon|^2}{2}+\frac{1}{2}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2+\frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right]\\ &=\frac{1}{r^{N-2}}\int_{\partial B(x,r)}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2+\frac{1}{r^{N-1}}\int_{B(x,r)}\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2},\end{aligned}$$

which yields the result.

### 2.3 Consequences for $\mu_*$

In this section, we will already derive two important consequences of the monotonicity formula. The first one can be directly translated into a property of the defect measure $\mu_*$. Let us first recall or set some general definitions about densities of measures.

**Definition 23** *Let $\nu$ be a non-negative Radon measure in $\mathbb{R}^N$. For $m \in \mathbb{N}$, the* $m$-dimensional lower density *of $\nu$ at the point $x$ is defined by*

$$\Theta_{*,m}(\nu,x)=\liminf_{r\to 0}\frac{\nu(B(x,r))}{\omega_m r^m},$$

*where $\omega_m$ denotes the volume of the unit ball $B^m$. Similarly, the* $m$-dimensional upper density $\Theta^*_m(\nu^t,x)$ *is given by*

$$\Theta^*_m(\nu,x)=\limsup_{r\to 0}\frac{\nu(B(x,r))}{\omega_m r^m}.$$

*When both quantities coincide, $\nu$ admits a* $m$-dimensional density $\Theta_m(\nu,x)$ *at the point $x$, defined as the common value.*

**Proposition 24** *Let $K\subset\Omega$ be any compact set and assume that $(H_0)$ holds. Then, there exists a constant $C(K)$ depending only on $K$ such that*

$$\Theta^*_{N-2}(\mu_*,x)\le C(K)M_0 \qquad \text{for all } x\in K. \tag{7}$$

*Proof.* Let $x\in K$ and $R=\mathrm{dist}(x,\partial\Omega)$. For $r<R$, we have by the monotonicity formula:

$$\frac{1}{r^{N-2}}\int_{B(x,r)}e_\varepsilon(u_\varepsilon)\le\frac{1}{R^{N-2}}\int_{B(x,R)}e_\varepsilon(u_\varepsilon)\le\mathrm{dist}(K,\partial\Omega)^{2-N}M_0|\log\varepsilon|.$$

Dividing by $|\log\varepsilon|$ and letting $\varepsilon$ go to zero we obtain

$$\frac{\mu_*(B(x,r))}{\omega_{N-2}r^{N-2}}\le\frac{\mathrm{dist}(K,\partial\Omega)^{2-N}}{\omega_{N-2}}M_0.$$

The conclusion follows taking the limsup as $r$ goes to zero.

*Remark 1.* i) Notice that the constant appearing in the previous proposition diverges as $K$ approaches the boundary of $\Omega$. This is unavoidable since we have not imposed any condition on the boundary of $\Omega$. ii) It also immediately follows from the monotonicity formula that the lower and upper $N-2$ densities coincide, i.e. that $\mu_*$ has a $N-2$ dimensional density everywhere in $\Omega$. We will come back to this later, when dealing with regularity properties.

**Comment.** Geometrically speaking, the bounds on the $N-2$ dimensional density of $\mu_*$ shows that this measure cannot be concentrated on objects of lower dimension than $N-2$. For example, for $N=3$ the measure $\mu_*$ cannot charge a point (as it is the case in dimension 2 [2]). There is still a wide variety of objects ranging between the dimensions $N-2$ and $N$; our subsequent analysis will restrict the possible situations much more!

The next consequence of the monotonicity formula, which we will derive now, deals with estimates on the potential part of the energy. It is a new estimate, which we will use in a crucial way in Section 4. For later convenience, we now fix the notation for the potential and define

$$V_\varepsilon(u_\varepsilon) = \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}.$$

**Proposition 25** *Assume that $u_\varepsilon$ solves $(GL)_\varepsilon$ on $\Omega$. Let $x \in \Omega$ and $0 < R_0 < R_1$ such that $B(x,R_1) \subset \Omega$. There exists $r(x) \in (R_0,R_1)$ such that*

$$\int_{B(x,r(x))} V_\varepsilon(u_\varepsilon) \le \frac{1}{\log(R_1/R_0)} \log\left(\frac{\tilde{E}_\varepsilon(u_\varepsilon,x,R_1)}{\tilde{E}_\varepsilon(u_\varepsilon,x,R_0)}\right) \int_{B(x,r(x))} e_\varepsilon(u_\varepsilon). \quad (8)$$

*Proof.* Define $f(r) = \tilde{E}_\varepsilon(u_\varepsilon,x,r)$ and $g(r) = r^{2-N}\int_{B(x,r)} V_\varepsilon(u_\varepsilon)$. It follows from the monotonicity formula that

$$\frac{d}{dr} f(r) \ge \frac{1}{r} g(r) \qquad \text{for all } r \in (R_0,R_1). \quad (9)$$

Let $F$ and $G$ be defined by the change of variable $f(r) = F(\log(r))$ and $g(r) = G(\log(r))$. Inequality (9) is transported into the inequality

$$\frac{d}{ds} F(s) \ge G(s) \qquad \text{for all } s \in (\log(R_0), \log(R_1)).$$

It suffices then to infer the next ODE lemma and the conclusion follows.

**Lemma 2** *Let $F$ and $G$ be absolutely continuous non-negative functions on the interval $[a,b] \subset \mathbb{R}$. Assume that $F' \ge G$ a.e. on $[a,b]$, then there exists $c \in (a,b)$ such that*

$$G(c) \le \frac{1}{b-a} \log\left(\frac{F(b)}{F(a)}\right) F(c).$$

*Proof.* Let $\lambda = \frac{1}{b-a}\log\left(\frac{F(b)}{F(a)}\right)$. If $G(s) > \lambda F(s)$ everywhere in $(a,b)$, and since by assumption $F' \geq G$, we obtain $F' > \lambda F$ so that

$$\frac{d}{ds}\left(\exp(-\lambda s)F(s)\right) > 0 \qquad \text{on } (a,b).$$

Integrating the previous inequality on $(a,b)$ we are led to

$$F(b) > \exp(-\lambda(a-b))F(a) = F(b),$$

a contradiction.

**Comment.** A particularly useful choice of $R_0$ and $R_1$ in Proposition 25 is given by $R_0 = \varepsilon^\alpha$ and $R_1 = \varepsilon^\beta$ for some $0 \leq \beta < \alpha < 1$. Indeed, for this choice we have $\log(R_1/R_0) \simeq |\log\varepsilon|$, and assuming that $\tilde{E}_\varepsilon(u_\varepsilon, x, \varepsilon^\alpha)$ and $\tilde{E}_\varepsilon(u_\varepsilon, x, \varepsilon^\beta)$ are of the same order we deduce from (8) that the potential energy on $B(x, r(x))$ is $|\log\varepsilon|$ times smaller than the total energy on the same ball.

# 3 Regularity theorems for $(\mathrm{GL})_\varepsilon$

## 3.1 Bochner, small energy regularity and clearing-out

A major difficulty in proving regularity theorems for $(\mathrm{GL})_\varepsilon$ stems from the factor $\varepsilon^{-2}$ which appears in the equation. A key ingredient in this viewpoint is given by the following Bochner identity:

**Proposition 26 (Bochner's identity)** *If $u_\varepsilon$ satisfies $(GL)_\varepsilon$, then the energy density $e_\varepsilon \equiv e_\varepsilon(u_\varepsilon)$ verifies*

$$-\Delta e_\varepsilon = -|D^2 u_\varepsilon| - |\Delta u_\varepsilon|^2 + \frac{1-|u_\varepsilon|^2}{\varepsilon^2}|\nabla u_\varepsilon|^2 - \frac{2}{\varepsilon^2}|u_\varepsilon|^2|\nabla|u_\varepsilon||^2,$$

*so that in particular*

$$-\Delta e_\varepsilon \leq C e_\varepsilon^2. \tag{10}$$

*Proof.* We have, since $u_\varepsilon$ verifies $(\mathrm{GL})_\varepsilon$,

$$-\Delta\left(\frac{|\nabla u_\varepsilon|^2}{2}\right) = -\Delta(\nabla u_\varepsilon)\cdot\nabla u_\varepsilon - |D^2u_\varepsilon|^2 = -\nabla(V_\varepsilon'(u_\varepsilon))\cdot\nabla u_\varepsilon - |D^2u_\varepsilon|^2$$

and

$$-\Delta\left(V_\varepsilon(u_\varepsilon)\right) = -\nabla(V_\varepsilon'(u_\varepsilon))\cdot\nabla u_\varepsilon - V_\varepsilon'(u_\varepsilon)\Delta u_\varepsilon = -\nabla(V_\varepsilon'(u_\varepsilon))\cdot\nabla u_\varepsilon - |\Delta u_\varepsilon|^2.$$

Bochner's identity follows by addition.

Note that (10) is not far from saying that $e_\varepsilon$ is subharmonic, in which case regularity estimates would follow very easily. The following theorem is an important consequence of the Bochner identity, it was first proved by R. Schoen for harmonic maps (see [8, 6] for proofs in a very similar context).

**Theorem 27 (Small energy regularity)** *Let $v_\epsilon$ be a solution of $(GL)_\varepsilon$ on the ball $B(R)$ for some $R > 0$. There exists a constant $\gamma_0 > 0$, depending only on $N$ such that if $R > \sqrt{\varepsilon}$ and if*

$$\frac{1}{R^{N-2}} \int_{B(R)} e_\epsilon(v_\epsilon) \leq \gamma_0,$$

*then*

$$\sup_{x \in B(R/2)} e_\epsilon(v_\epsilon)(x) \leq \frac{C}{R^N} \int_{B(R)} e_\epsilon(v_\epsilon) \tag{11}$$

*where the constant $C$ depends only on $N$.*

In general, an estimate like (11) does not hold everywhere for solutions of $(\mathrm{GL})_\varepsilon$, in particular not in the so called "vorticity set" where $|u_\varepsilon| \leq 1/2$. The following theorem will therefore play a central role (see [3] for the proof, which uses the monotonicity formula in a crucial way):

**Theorem 28 (Clearing-out)** *Let $0 < \sigma \leq \frac{1}{2}$ be given. There exist a constant $\eta_0 > 0$ depending only on $\sigma$ and $N$ such that if $u_\varepsilon$ verifies $(GL)_\varepsilon$ on $B(R)$, $R > \sqrt{\varepsilon}$ and if*

$$\frac{1}{R^{N-2}} \int_{B(R)} e_\varepsilon(u_\varepsilon) \leq \eta_0 |\log \varepsilon|,$$

*then*

$$|u_\varepsilon| \geq 1 - \sigma \qquad \text{on} \quad B(\frac{R}{2}).$$

In the next proposition, we prove that away from the vorticity set an estimate like (11) holds.

**Proposition 29** *Let $u_\varepsilon$ be a solution of $(GL)_\varepsilon$ on $B(R)$ such that $u_\varepsilon$ verifies assumption $(H_0)$. There exist a constant $0 < \sigma \leq \frac{1}{2}$ depending only on $N$ such that if $R > \sqrt{\varepsilon}$ and if*

$$|u_\varepsilon| \geq 1 - \sigma \qquad \text{on } B(R),$$

*then*

$$\sup_{x \in B(R/2)} e_\varepsilon(u_\varepsilon)(x) \leq \frac{C}{R^N} \int_{B(R)} e_\varepsilon(u_\varepsilon),$$

*where the constant $C$ depends only on $M_0$ and $N$.*

*Proof.* Using the scaling properties of $(\mathrm{GL})_\varepsilon$, we may assume without loss of generality that $R = 1$. Since by assumption $|u_\varepsilon| \geq 1 - \sigma \geq \frac{1}{2}$ on $B(1)$, there is some real-value function $\varphi_\varepsilon$ defined on $B(1)$ such that

$$u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \qquad \text{in } B(1), \tag{12}$$

where $\rho_\varepsilon = |u_\varepsilon|$. Changing $u_\varepsilon$ possibly by a constant phase, we may impose the additional condition

$$\frac{1}{|B(1)|} \int_{B(1)} \varphi_\varepsilon = 0. \tag{13}$$

We split as previously the estimates for the phase $\varphi_\varepsilon$ and for the modulus $\rho_\varepsilon$, and we begin with the phase. Inserting (12) into $(\mathrm{GL})_\varepsilon$ we are led to the elliptic equation

$$-\mathrm{div}(\rho_\varepsilon \nabla \varphi_\varepsilon) = 0 \qquad \text{in } B(1). \tag{14}$$

In contrast with the equation for the modulus which we will tackle later, (14) has the advantage that the explicit dependence on $\varepsilon$ has been removed. We will handle (14) as a linear equation for the function $\varphi_\varepsilon$, $\rho_\varepsilon$ being considered as a coefficient. In the sequel, we write $\varphi = \varphi_\varepsilon$ and $\rho = \rho_\varepsilon$ when this is not misleading. In order to avoid boundary conditions, we consider the truncated function $\tilde{\varphi}$ defined by $\tilde{\varphi} = \varphi\, \chi$, where $\chi$ is a smooth cut-off function such that

$$\chi \equiv 1 \text{ on } B(\frac{4}{5}) \qquad \text{and} \qquad \chi \equiv 0 \text{ on } \mathbb{R}^N \setminus B(\frac{5}{6}).$$

The function $\tilde{\varphi}$ then verifies the equation

$$-\mathrm{div}(\rho \nabla \tilde{\varphi}) = \mathrm{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi \qquad \text{in } B(1). \tag{15}$$

Moreover, by construction

$$\mathrm{supp}(\tilde{\varphi}) \subset B(\frac{4}{5}).$$

Since by assumption $\rho$ is close to 1, it is natural to treat the l.h.s. of (15) as a perturbation of the Laplace operator, and to rewrite (15) as follows

$$-\Delta \tilde{\varphi} = \mathrm{div}((\rho^2 - 1)\nabla \tilde{\varphi}) + \mathrm{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi \qquad \text{in } B(1).$$

We introduce the function $\varphi_0$ defined on $B(1)$ as the solution of

$$\begin{cases} -\Delta \varphi_0 = \mathrm{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi & \text{in } B(1), \\ \quad \varphi_0 = 0 & \text{on } \partial B(1). \end{cases} \tag{16}$$

In particular, since $\chi \equiv 1$ on $B(\frac{4}{5})$, we have

$$-\Delta \varphi_0 = 0 \qquad \text{in } B(\frac{4}{5}). \tag{17}$$

We set $\varphi_1 = \tilde{\varphi} - \varphi_0$, i.e.

$$\tilde{\varphi} = \varphi_0 + \varphi_1.$$

We will show that $\varphi_1$ is essentially a perturbation term.

At this stage, we divide the estimates into several steps. We start with linear estimates for $\varphi_0$.

**Step 1: Estimates for $\boldsymbol{\varphi_0}$.** We claim that

$$\|\nabla\varphi_0\|^2_{L^{2^*}(B(1))} \le C_1 \left[\int_{B(1)} e_\varepsilon(u_\varepsilon)\right] \tag{18}$$

and

$$\|\nabla\varphi_0\|^2_{L^\infty(B(3/4))} \le C_2 \left[\int_{B(1)} e_\varepsilon(u_\varepsilon)\right], \tag{19}$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent in dimension $N$.

*Proof.* The first estimate follows from the linear theory for the Laplace operator, whereas the second follows from the first one and the fact that $\varphi_0$ is harmonic on $B(4/5)$.

**Step 2: The equation for $\boldsymbol{\varphi_1}$.** The function $\varphi_1$ verifies the elliptic problem

$$\begin{cases} -\Delta\varphi_1 = \operatorname{div}((\rho^2-1)\nabla\tilde{\varphi}) & \text{in } B(1), \\ \quad\ \varphi_1 = 0 & \text{on } \partial B(1). \end{cases} \tag{20}$$

It is convenient to rewrite equation (20) as

$$-\Delta\varphi_1 = \operatorname{div}((\rho^2-1)\nabla\varphi_1) + \operatorname{div}(g_0), \tag{21}$$

where we have set $g_0 = (\rho^2-1)\nabla\varphi_0$. Using (18), we obtain, for any $2 \le q < 2^*$, the estimate for $g_0$

$$\|g_0\|^q_{L^q(B(1))} \le C\,(M_0|\log\varepsilon|)^{\frac{2^*-q}{2^*}}\,\varepsilon^{\frac{2(2^*-q)}{2^*}}\,\|e_\varepsilon(u_\varepsilon)\|^{\frac{q}{2}}_{L^1}. \tag{22}$$

Indeed,

$$\begin{aligned}\int_{B(1)} |\rho^2-1|^q|\nabla\varphi_0|^q &\le \left(\int_{B(1)} |\nabla\varphi_0|^{2^*}\right)^{\frac{q}{2^*}} \left(\int_{B(1)} |\rho^2-1|^{\frac{2^*q}{2^*-q}}\right)^{\frac{2^*-q}{2^*}} \\ &\le C\|e_\varepsilon(u_\varepsilon)\|^{\frac{q}{2}}_{L^1}\varepsilon^{\frac{2(2^*-q)}{2^*}} \left(\int_{B(1)} \frac{|\rho^2-1|^2}{4\varepsilon^2}\right)^{\frac{2^*-q}{2^*}},\end{aligned}$$

and the conclusion follows.

We now estimate $\varphi_1$ from (21) through a fixed point argument.

**Step 3: The fixed point argument.** Equation (21) may be rewritten as

$$\varphi_1 = \mathcal{T}(\operatorname{div}((\rho^2-1)\nabla\varphi_1)) + \mathcal{T}(\operatorname{div} g_0),$$

which is of the form

$$(\mathrm{Id} - A)\varphi_1 = b$$

where $\mathcal{T} = \Delta^{-1}$, $A$ is the linear operator $v \mapsto \mathcal{T}(\operatorname{div}((\rho^2-1)\nabla v))$ and $b = \mathcal{T}(\operatorname{div} g_0)$. Consider the Banach space $X_q = W_0^{1,q}(B(1))$. It follows from the linear theory for $\mathcal{T}$ that $A : X_q \to X_q$ is linear continuous and that

$$\|A\|_{\mathcal{L}(X_q)} \leq C(q)\|1-\rho\|_{L^\infty(B(1))}.$$

In particular, we may fix $\sigma > 0$ such that

$$C(q)\|1-\rho\|_{L^\infty(B(1))} \leq C(q)\sigma < \frac{1}{2}.$$

With this choice of $\sigma$, we deduce that $I - A$ is invertible on $X_q$ and

$$\|\varphi_1\|_{X_q} \leq C\|b\|_{X_q}. \tag{23}$$

Finally, by (22) we obtain

$$\|b\|_{X_q} = \|\mathcal{T}(\operatorname{div} g_0)\|_{X_q} \leq C\|g_0\|_{L^q} \leq C\,\varepsilon^{\frac{2(2^*-q)}{q2^*}}\,\|e_\varepsilon(u_\varepsilon)\|_{L^1}^{\frac{1}{2}+\frac{2^*-q}{q2^*}}.$$

The following estimate for $\varphi_1$ then follows from (23)

$$\|\nabla\varphi_1\|_{L^q(B(1))} \leq C\,(M_0|\log\varepsilon|)^{\frac{2^*-q}{q2^*}}\,\varepsilon^{\frac{2(2^*-q)}{q2^*}}\,\|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))}^{\frac{1}{2}}. \tag{24}$$

We now combine the estimates for $\varphi_0$ and $\varphi_1$.

**Step 4: Improved integrability of $\nabla\tilde{\varphi}$.** Combining (18) and (24) we obtain

$$\|\nabla\tilde{\varphi}\|_{L^q(B(1))} \leq C(q, M_0)\|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))}^{\frac{1}{2}}, \tag{25}$$

where $C(q, M_0)$ depends only on $q$ and $M_0$.

**Comment.** Since $q > 2$, the previous estimate presents a substantial improvement over the corresponding inequality with $q$ replaced by 2, which follows directly from $(H_0)$. This improvement is crucial in order to prove the smallness of both the modulus and potential terms in the energy, which we derive now.

**Step 5: Estimates for the modulus and potential terms.**

The function $\rho$ satisfies the equation

$$-\Delta\rho + \rho|\nabla\varphi|^2 = \rho\frac{(1-\rho^2)}{\varepsilon^2}. \tag{26}$$

Since $\chi \equiv 1$ on $B(\frac{4}{5})$, we have $\varphi = \tilde{\varphi}$ on $B(\frac{4}{5})$. Let $\xi$ be a non-negative cut-off function such that $\xi \equiv 1$ on $B(\frac{3}{4})$ and $\xi \equiv 0$ outside $B(\frac{4}{5})$. Multiplying (26) by $(1-\rho^2)\xi$ and integrating by parts we obtain

$$\int_{B(1)} 2\rho|\nabla\rho|^2\xi + \int_{B(1)} \rho\frac{(1-\rho^2)^2}{\varepsilon^2} = \int_{B(1)} \nabla\rho\cdot\nabla\xi(1-\rho^2) + \int_{B(1)} \rho(1-\rho^2)|\nabla\tilde{\varphi}|^2\xi.$$

Hence, since $\rho \geq \frac{1}{2}$ on $B(1)$ we obtain

$$\begin{aligned}\int_{B(3/4)} |\nabla\rho|^2 + V_\varepsilon(u_\varepsilon) \leq C\varepsilon \left(\int_{B(1)} |\nabla\rho|^2\right)^{\frac{1}{2}} \left(\int_{B(1)} V_\varepsilon(u_\varepsilon)\right)^{\frac{1}{2}} \\ + C\left(\int_{B(\frac{4}{5})} |\nabla\tilde{\varphi}|^q\right)^{\frac{2}{q}} \left(\int_{B(\frac{4}{5})} (1-\rho^2)^{\frac{q}{q-2}}\right)^{\frac{q-2}{q}}\end{aligned}$$

so that using (25) we finally infer that

$$\int_{B(3/4)} [\,\big|\nabla|u_\varepsilon|\big|^2 + V_\varepsilon(u_\varepsilon)] \leq C(M_0)\varepsilon^\alpha \int_{B(1)} e_\varepsilon(u_\varepsilon), \tag{27}$$

for some $\alpha > 0$ [here we have fixed $2 < q < 2^*$].

To summarize, we have proved at this stage that

$$e_\varepsilon(u_\varepsilon) \leq |\nabla\varphi_0|^2 + r_\varepsilon, \tag{28}$$

for some $r_\varepsilon \geq 0$ which verifies

$$\int_{B(3/4)} r_\varepsilon \leq C(M_0)\varepsilon^\alpha \int_{B(1)} e_\varepsilon(u_\varepsilon), \tag{29}$$

for some small $\alpha > 0$ depending only on $N$. Therefore, we set

**Step 6: Proof of Proposition 29 completed.**

We are going to apply Theorem 27 to a suitably scaled version of $u_\varepsilon$.

Let $\sqrt{\varepsilon} < r_0 < \frac{1}{8}$, to be determined later, set $\epsilon = \frac{\varepsilon}{r_0}$ and let $x_0 \in B(1/2)$ be fixed. Consider the map $v_\epsilon$ defined on $B(1)$ by

$$v_\epsilon(x) = u_\varepsilon\left(\frac{x - x_0}{r_0}\right)$$

so that $u_\varepsilon(x_0) = v_\epsilon(0)$. By scaling, we have

$$\int_{B(1)} e_\epsilon(v_\epsilon) = \frac{1}{r_0^{N-2}} \int_{B(x_0,r_0)} e_\varepsilon(u_\varepsilon). \tag{30}$$

Note in particular, since $r_0 < \frac{1}{8}$, that $B(x_0, r_0) \subset B(3/4)$, and we may apply the decomposition (28) to assert that

$$\int_{B(x_0,r_0)} e_\varepsilon(u_\varepsilon) \leq \text{meas}(B(x_0,r_0)) \cdot \|\nabla\varphi_0\|^2_{L^\infty} + \int_{B(3/4)} r_\varepsilon$$
$$\leq C r_0^N \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))} + C(M_0)\varepsilon^\alpha \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))}.$$

Hence, going back to (30)

$$\int_{B(1)} e_\epsilon(v_\epsilon) \leq C r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))} + C(M_0) r_0^{2-N} \varepsilon^\alpha \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))}. \quad (31)$$

Therefore, we choose

$$r_0 = \inf\left\{\frac{1}{8}, (\frac{2C\|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))}}{\gamma_0})^{-\frac{1}{2}}\right\}.$$

Note in particular that $r_0^{2-N}$ diverges at most as $|\log\varepsilon|^{\frac{N-2}{2}}$. Hence, for $\varepsilon$ sufficiently small,

$$C(M_0) r_0^{2-N} \varepsilon^\alpha \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))} \leq \frac{\gamma_0}{2}.$$

On the other hand, by construction,

$$\omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} \leq \frac{\gamma_0}{2}.$$

Applying Theorem 27 to $v_\epsilon$ for $R = 1$, we therefore deduce that

$$r_0^2 e_\varepsilon(u_\varepsilon)(x_0) = e_\epsilon(v_\epsilon)(0) \leq C \int_{B(1)} e_\epsilon(v_\epsilon)$$
$$\leq C(M_0) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))} + C(M_0) r_0^{2-N} \varepsilon^\alpha \|e_\varepsilon(u_\varepsilon)\|_{L^1(B(1))},$$

so that for $\varepsilon$ sufficiently small

$$e_\varepsilon(u_\varepsilon)(x_0) \leq C(M_0) \int_{B(1)} e_\varepsilon(u_\varepsilon),$$

and the proof is complete.

Combining Theorem 27 with Proposition 29 we derive

**Theorem 30** *Assume that $u_\varepsilon$ is a solution of $(GL)_\varepsilon$ on $B(x,R)$ and that $(H_0)$ holds. There exist a constant $\eta_1$ depending only on $N$ and $M_0$ such that if $R > \sqrt{\varepsilon}$ and if*

$$\frac{1}{R^{N-2}} \int_{B(R)} e_\varepsilon(u_\varepsilon) \leq \eta_1 |\log\varepsilon|$$

*then*

$$e_\varepsilon(u_\varepsilon) \leq \frac{C}{R^N} \int_{B(R)} e_\varepsilon(u_\varepsilon) \qquad \text{on } B(\frac{R}{4}),$$

*where $C$ depends only on $N$ and $M_0$.*

### 3.2 Consequences for $\mu_*$

We are now in position to obtain much more information on the concentration measure $\mu_*$. Recall that we already know that the $N-2$ dimensional density of $\mu_*$ is locally bounded.

**Proposition 31** *There exists $\eta > 0$ such that for every $x \in \Omega$,*

$$\text{either} \qquad \Theta_{N-2}(\mu_*, x) \geq \eta \qquad \text{or} \qquad \Theta_{N-2}(\mu_*, x) = 0.$$

*Moreover,*

$$\text{if} \qquad \Theta_{N-2}(\mu_*, x) = 0 \qquad \text{then} \qquad \Theta^*_N(\mu_*, x) < +\infty.$$

*Proof.* Define $\eta = \frac{\eta_1}{\omega_N}$. Assume that $x \in \Omega$ is such that $\Theta_{N-2}(\mu_*, x) < \eta$. By definition, there exists $R > 0$ (which we can assume small enough so that $B(x,R) \subset \Omega$) such that

$$\frac{\mu_*(B(x,R))}{\omega_N R^{N-2}} < \eta.$$

Hence, for every $\varepsilon$ sufficiently small

$$\frac{1}{R^{N-2}} \int_{B(x,R)} e_\varepsilon(u_\varepsilon) < \eta_1 |\log \varepsilon|.$$

Applying Theorem 30 we obtain

$$e_\varepsilon(u_\varepsilon) \leq C(M_0) \frac{1}{R^N} \int_{B(x,R)} e_\varepsilon(u_\varepsilon) < C(M_0) R^{-2} \eta_1 |\log \varepsilon|$$

on $B(x, R/4)$. In particular, for any $r < R/4$

$$\mu_*(B(x,r)) \leq r^N R^{-2} \eta_1$$

and the conclusion follows taking the limsup as $r$ goes to zero.

In view of the previous proposition, we introduce the concentration set

$$\Sigma_\mu := \left\{ x \in \Omega \text{ s.t. } \Theta_{N-2}(\mu_*, x) > 0 \right\}.$$

**Proposition 32** *The concentration set $\Sigma_\mu$ is closed in $\Omega$ and of locally finite $\mathcal{H}^{N-2}$ Hausdorff measure.*

*Proof.* Since by the previous proposition

$$\Sigma_\mu := \left\{ x \in \Omega \text{ s.t. } \Theta_{N-2}(\mu_*, x) \geq \eta \right\},$$

its closedness follows from the upper semi-continuity of the density. Now, let $K \subset \Omega$ be any compact subset and set $\delta_0 = \mathrm{dist}(K, \partial\Omega)$. Fix $\delta < \delta_0/2$ and consider a standard (say parallepipedic) covering of $\mathbb{R}^N$ such that

$$\mathbb{R}^N \subseteq \cup_{j\in I} B(x_j,\delta), \qquad \text{and} \qquad B(x_i,\frac{\delta}{2}) \cap B(x_j,\frac{\delta}{2}) = \emptyset \text{ for } i \neq j.$$

Set

$$I_\delta = \{i \text{ s.t. } B(x_i,\delta) \cap \Sigma_\mu \neq \emptyset\}.$$

For $i \in I_\delta$, there exists some $y_i \in \Sigma_\mu \cap B(x_i,\delta)$. Hence, by Proposition 31,

$$\mu_*(B(y_i,\delta) > \eta\omega_N \delta^{2-N},$$

and in particular

$$\mu_*(B(x_i,(2\delta)) > \eta\omega_N \delta^{2-N}. \tag{32}$$

On the other hand, since the balls $B(x_i,\frac{\delta}{2})$ are disjoints, the balls $B(x_i,2\delta)$ cover at most $C$ times $\mathbb{R}^N$, where $C$ is a constant depending only on $N$. Therefore,

$$\sum_{i\in I_\delta} \mu_*(B(x_i,2\delta)) \leq CM_0. \tag{33}$$

Combining (32) and (33) we obtain

$$\sharp I_\delta \leq CM_0\delta^{2-N}.$$

Since by definition, $\mathcal{H}^{N-2}(\Sigma_\mu) \leq C \limsup_{\delta\to 0}(\sharp I_\delta)\delta^{N-2}$, the conclusion follows.

We may now state and prove our main structure and regularity theorem concerning the measure $\mu_*$ and its concentration set $\Sigma_\mu$. We first recall some the notion of rectifiability.

**Definition 33** *A set $\Sigma \subset \mathbb{R}^N$ is said to be* $m$-rectifiable *($0 \leq m \leq N$) if $\mathcal{H}^m$ almost all of $\Sigma$ can be covered by the union of countably many Lipschitz images of $B^m$.*

*A Radon measure $\nu$ on $\mathbb{R}^N$ is said to be* $m$-rectifiable *if there exist an $m$-rectifiable set $\Sigma$ and a nonnegative density function $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathcal{H}^m \llcorner \Sigma)$ such that*

$$\nu = h(\cdot)\,\mathcal{H}^m \llcorner \Sigma.$$

**Theorem 34 (Structure and regularity)** *The set $\Sigma_\mu$ is $(N-2)$-rectifiable and the measure $\mu_*$ can be exactly decomposed as*

$$\mu_* = g(.)\mathcal{H}^N + \Theta_*(.)\mathcal{H}^{N-2} \llcorner \Sigma_\mu,$$

*where $g \in L^\infty_{\mathrm{loc}}(\Omega)$ and the density function $\Theta_* \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{H}^{N-2} \llcorner \Sigma_\mu)$.*

*Proof.* The rectifiability of $\Sigma_\mu$ follows from the lower bound on the density given in Proposition 31 and Preiss regularity theorem [7].

Since $\Sigma_\mu$ is closed,

$$\mu_* = \mu_* \llcorner (\Omega \setminus \Sigma_\mu) + \mu_* \llcorner \Sigma_\mu.$$

By propositions 31 and 32, $\mu_* \llcorner \Sigma_\mu$ is absolutely continuous with respect to $\mathcal{H}^{N-2} \llcorner \Sigma_\mu$, and of (locally) finite $\mathcal{H}^{N-2}$ measure. We therefore infer from the Radon-Nikodym theorem that

$$\mu_* \llcorner \Sigma_\mu = \Theta_*(.)\mathcal{H}^{N-2} \llcorner \Sigma_\mu,$$

where $\theta_*$ is the Radon-Nikodym derivative.

In the same way, we obtain

$$\mu_* \llcorner (\Omega \setminus \Sigma_\mu) = g(.)\mathcal{H}^N,$$

and the proof is complete.

*Remark 2.* Notice that working a little more the proof of Proposition 29, we could state that actually $g(.) = |\nabla\Phi|^2(.)$ for some harmonic function $\Phi$ defined on $\Omega$.

## 4 Potential and modulus estimates

The energy $e_\varepsilon$ consists of two parts: the kinetic one, $|\nabla u_\varepsilon|^2/2$, and the potential one, $V_\varepsilon(u_\varepsilon)$. The kinetic energy arises from variations in both the phase and the modulus of $u_\varepsilon$. In this section, we will prove that under assumption $(H_0)$ only the gradient of the phase contributes to $\mu_*$, the modulus and potential parts remaining uniformly bounded. More precisely, we will prove the following theorems (which are new in dimension $N \geq 3$):

**Theorem 35** *Let $u_\varepsilon$ be a solution of $(GL)_\varepsilon$ on $\Omega$ such that assumption $(H_0)$ is verified. For any $K \subset \Omega$ compact, there exist a constant $C(K, M_0)$ depending only on $K$ and $M_0$ such that*

$$\int_K V_\varepsilon(u_\varepsilon) \leq C(K, M_0).$$

and

**Theorem 36** *Let $u_\varepsilon$ be a solution of $(GL)_\varepsilon$ on $\Omega$ such that assumption $(H_0)$ is verified. For any $K \subset \Omega$ compact, there exist a constant $C(K, M_0)$ depending only on $K$ and $M_0$ such that*

$$\int_K |\nabla|u_\varepsilon||^2 \leq C(K, M_0).$$

In the proofs of Theorems 35 and 36 we will need the following easy pointwise estimates, whose proofs are based on the maximum principle.

**Lemma 3** *Let $u_\varepsilon$ be a solution of $(GL)_\varepsilon$ on $\Omega = B(R)$, $R > \sqrt{\varepsilon}$. There exist a constant $C$ such that*

$$|u_\varepsilon|(x) \leq 1 + C\frac{\varepsilon^2}{\mathrm{dist}(x, \partial\Omega)}.$$

*Moreover, if $|u_\varepsilon| \geq \frac{1}{2}$ on $B(R)$, then*

$$|u_\varepsilon| \geq 1 - C\frac{\varepsilon^2}{R^2} - C\varepsilon^2 |\nabla\varphi_\varepsilon|^2_{L^\infty(B(R))}$$

*on $B(R/2)$, where $u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$.*

*Proof of Theorem 35.* Let $K \subset \Omega$ be given and $\delta = \mathrm{dist}(K, \partial\Omega)$. We decompose $K$ in two parts $K_1$ and $K_2$, where

$$K_1 = \left\{x \in K \quad \text{s.t.} \quad \mathrm{dist}(x, \{|u_\varepsilon| \leq 1 - \sigma\}) \leq \varepsilon^{\frac{1}{8}}\right\}$$

and $K_2 = K \setminus K_1$. If $x \in K_1$, we infer from Proposition 25 that there exists $r(x) \in (\varepsilon^{\frac{1}{16}}, \delta/2)$ such that

$$\int_{B(x,r(x))} V_\varepsilon(u_\varepsilon) \leq \frac{C}{|\log\varepsilon|} \log\left(\frac{\tilde{E}_\varepsilon(u_\varepsilon, x, \delta/2)}{\tilde{E}_\varepsilon(u_\varepsilon, x, \varepsilon^{\frac{1}{16}})}\right) \int_{B(x,r(x))} e_\varepsilon(u_\varepsilon). \tag{34}$$

On the other hand, by monotonicity and assumption $(H_0)$ we have

$$\tilde{E}_\varepsilon(u_\varepsilon, x, \delta/2) \leq C\delta^{2-N} M_0 |\log\varepsilon|, \tag{35}$$

and since $x \in K_1$, we also have

$$\tilde{E}_\varepsilon(u_\varepsilon, x, \varepsilon^{\frac{1}{16}}) \geq C\eta_0 |\log\varepsilon|. \tag{36}$$

Indeed, to obtain the last inequality observe that by assumption there exists $\tilde{x} \in \{|u_\varepsilon| \leq 1-\sigma\}$ such that $|x - \tilde{x}| \leq \varepsilon^{\frac{1}{8}}$. Therefore, applying the clearing-out theorem at the point $\tilde{x}$, we obtain

$$\tilde{E}_\varepsilon(u_\varepsilon, x, \varepsilon^{\frac{1}{16}}) \geq \left(\frac{\varepsilon^{\frac{1}{16}}}{\varepsilon^{\frac{1}{16}} - \varepsilon^{\frac{1}{8}}}\right)^{2-N} \tilde{E}_\varepsilon(u_\varepsilon, \tilde{x}, \varepsilon^{\frac{1}{16}} - \varepsilon^{\frac{1}{8}}) \geq \frac{\eta_0}{2}|\log\varepsilon|$$

for $\varepsilon$ sufficiently small.

Combining (34),(35) and (36) we are led to

$$\int_{B(x,r(x))} V_\varepsilon(u_\varepsilon) \leq \frac{C(M_0, K)}{|\log\varepsilon|} \int_{B(x,r(x))} e_\varepsilon(u_\varepsilon). \tag{37}$$

From the open covering $\{B(x, r(x))\}_{x\in K_1}$ of $K_1$, we may extract a Besicovitch sub-covering, so that by addition we finally deduce that

$$\int_{K_1} V_\varepsilon(u_\varepsilon) \le \frac{C(M_0,K)}{|\log\varepsilon|} \int_\Omega e_\varepsilon(u_\varepsilon) \le C(K,M_0). \tag{38}$$

We are left with the estimate on $K_2$. Let $x \in K_2$ and set $R = \varepsilon^{\frac{1}{8}}$. We deduce from Proposition 29 that

$$|\nabla\varphi_\varepsilon|^2_{L^\infty(B(x,R/2))} \le C(K)M_0\varepsilon^{-\frac{1}{4}}|\log\varepsilon|.$$

Hence, it follows from Lemma 3 that on $B(x,R/4)$,

$$1 - C(K)(1 + M_0|\log\varepsilon|)\varepsilon^{\frac{7}{4}} \le |u_\varepsilon| \le 1 + C(K)\varepsilon^2.$$

In particular

$$V_\varepsilon(u_\varepsilon) \le C(K,M_0)\varepsilon \qquad \text{in } K_2,$$

so that the conclusion follows by integration. □

*Proof of Theorem 36.* Let $K \subset \Omega$ be given and let $\chi$ be a cut-off function supported in $\Omega$ such that $\chi \equiv 1$ on $K$. Define $v_\varepsilon = \chi u_\varepsilon$. Then,

$$-\Delta v_\varepsilon = \frac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2)\chi - u_\varepsilon\Delta\chi - 2\nabla u_\varepsilon\cdot\nabla\chi \equiv f_\varepsilon.$$

Note that in view of Theorem 35,

$$\|f_\varepsilon\|_{L^2(\Omega)} \le C(K,M_0)\varepsilon^{-1}.$$

By elliptic regularity theory, we therefore deduce that

$$\|v_\varepsilon\|_{H^2(\Omega)} \le C(K,M_0)\varepsilon^{-1}.$$

On the other hand, since $|v_\varepsilon| \le C(K)$, by the Gagliardo-Nirenberg inequality

$$\|\nabla v_\varepsilon\|_{L^4(\Omega)} \le \|v_\varepsilon\|^{\frac{1}{2}}_{H^2(\Omega)}\|v_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\Omega)},$$

we infer from the previous estimate that

$$\|\nabla v_\varepsilon\|_{L^4(\Omega)} \le C(K,M_0)\varepsilon^{-\frac{1}{2}}. \tag{39}$$

The modulus $\rho \equiv |u_\varepsilon|$ verifies the equation

$$-\Delta(\rho^2) + 2|\nabla u_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho^2(1-\rho^2). \tag{40}$$

Multiplying (40) by $(1-\rho^2)\chi$ and integrating by parts we obtain

$$\begin{aligned}\int_K |\nabla(\rho^2)|^2 &\le 2\int_\Omega (1-\rho^2)\chi^2|\nabla u_\varepsilon|^2\\ &\le 2\int_\Omega(1-\rho^2)|\nabla v_\varepsilon|^2 + 2\int_\Omega(1-\rho^2)|\nabla\chi|^2|u_\varepsilon|^2\\ &\le \varepsilon\|\nabla v_\varepsilon\|^2_{L^4}\left(\int_{\mathrm{supp}\chi} V_\varepsilon(u_\varepsilon)\right)^{\frac{1}{2}} + C\varepsilon\left(\int_{\mathrm{supp}\chi} V_\varepsilon(u_\varepsilon)\right)^{\frac{1}{2}}\\ &\le C(K,M_0),\end{aligned}$$

where we have used Theorem 35 and (39). Finally,

$$\begin{aligned}\int_K |\nabla\rho|^2 &= \frac{1}{2}\int_K |\nabla(\rho^2)|^2 + \frac{1}{2}\int_K (1-\rho^2)|\nabla(\rho^2)|^2 \\ &\le C(K,M_0) + C\varepsilon\left(\int_K V_\varepsilon(u_\varepsilon)\right)^{\frac{1}{2}}\left(\int_K |\nabla u_\varepsilon|^4\right)^{\frac{1}{2}} \\ &\le C(K,M_0),\end{aligned}$$

and the proof is complete. □

# 5 $\Sigma_\mu$ is a minimal surface

## 5.1 Classical and weak notions of mean curvature

Let $\Sigma$ be a smooth compact manifold of dimension $k$, and $\gamma_0 : \Sigma \to \mathbb{R}^N$ ($N \geq k$) a smooth embedding, so that $\Sigma^0 = \gamma_0(\Sigma)$ is a smooth k-dimensional submanifold of $\mathbb{R}^N$. The mean curvature vector at the point $x$ of $\Sigma^0$ is the vector of the orthogonal space $(T_x\Sigma^0)^\perp$ given by

$$\mathbf{H}_{\Sigma^0}(x) = -\sum_{\alpha=1}^{N-k}\left(\sum_{j=1}^{k}(\tau_j \cdot \frac{\partial \nu^\alpha}{\partial \tau_j})\nu^\alpha\right) = -\sum_{\alpha=1}^{N-k}(\mathrm{div}_{T_x\Sigma^0}\nu^\alpha)\,\nu^\alpha, \tag{41}$$

where $(\tau_1, ..., \tau_k)$ is an orthonormal moving frame on $T_x\Sigma^0$, $(\nu^1, ..., \nu^{N-k})$ is an orthonormal moving frame on $(T_x\Sigma^0)^\perp$, and $\mathrm{div}_{T_x\Sigma^0}$ denotes the tangential divergence at the point $x$. The integral formulation of (41) is given by

$$\int_{\Sigma^0} \mathrm{div}_{T_x\Sigma^0}\mathbf{X}\, d\mathcal{H}^k = -\int_{\Sigma^0}\mathbf{H}_{\Sigma^0}\cdot\mathbf{X}\, d\mathcal{H}^k, \tag{42}$$

for all $\mathbf{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. The vectors $\mathbf{H}_{\Sigma^0}(\cdot)$ are uniquely determined by (42), and in particular the definition in (41) does not depend on the choice of orthonormal frames.

Assume now that $\Sigma$ is only rectifiable. Then for $\mathcal{H}^k$-a.e. $x \in \Sigma$, there exist a unique tangent space $T_x\Sigma$ belonging to the Grassmanian $G_{N,k}$. The distributional first variation of $\nu$ is the vector-valued distribution $\delta\nu$ defined by

$$\delta\nu(\mathbf{X}) = \int_\Sigma \mathrm{div}_{T_x\Sigma}\mathbf{X}d\nu \qquad \text{for all } \mathbf{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N). \tag{43}$$

In case $|\delta\nu|$ is a measure absolutely continuous with respect to $\nu$, we say that $\nu$ has a first variation and we may write

$$\delta\nu = \mathbf{H}\nu,$$

where $\mathbf{H}$ is the Radon-Nikodym derivative of $\delta\nu$ with respect to $\nu$. In this case, formula (43) becomes

$$\int_{\Sigma} \mathrm{div}_{T_x\Sigma} \mathbf{X} d\nu = -\int_{\Sigma} \mathbf{H} \cdot \mathbf{X}\, d\nu. \tag{44}$$

*Remark 3.* Notice that in the smooth case, this notion coincides with the definition (41), in view of (42). Notice also that the component of $\mathbf{H}$ which is orthogonal to $T_x\Sigma$ is independent of the density $\Theta$. However, if $\Theta$ is non constant, then $\mathbf{H}$ may have a tangential part.

### 5.2 Deriving the curvature equation

Let $\mathbf{X} \in \mathcal{D}(\Omega, \mathbb{R}^N)$ be a smooth vector field. We have

$$\begin{aligned}\int_{\Omega} e_\varepsilon(u_\varepsilon)\mathrm{div}\mathbf{X} &= -\int_{\Omega} \nabla e_\varepsilon(u_\varepsilon)\cdot \mathbf{X} \\ &= -\int_{\Omega} \frac{1}{2}\nabla(|\nabla u_\varepsilon|^2) + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)(-2u_\varepsilon \nabla u_\varepsilon)\cdot \mathbf{X},\end{aligned} \tag{45}$$

and

$$\begin{aligned}\int_{\Omega}\sum_{i,j}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial X^i}{\partial x_j} &= -\int_{\Omega}\sum_{i,j}(\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\frac{\partial u_\varepsilon}{\partial x_j} + \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial^2 u_\varepsilon}{\partial x_j^2})X^i \\ &= -\int_{\Omega}\nabla u_\varepsilon \cdot \mathbf{X}\Delta u_\varepsilon - \int_{\Omega}\sum_{i,j}\frac{\partial}{\partial x_i}\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^2 X^i \\ &= -\int_{\Omega}\nabla u_\varepsilon \cdot \mathbf{X}\Delta u_\varepsilon - \int_{\Omega}\frac{1}{2}\nabla(|\nabla u_\varepsilon|^2)\cdot \mathbf{X}.\end{aligned} \tag{46}$$

Since $u_\varepsilon$ is a solution of $(\mathrm{GL})_\varepsilon$, we deduce from (45) and (46) that

$$\begin{aligned}&\frac{1}{|\log\varepsilon|}\int_{\Omega}\left(e_\varepsilon(u_\varepsilon)\delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial X^i}{\partial x_j} \\ &\qquad = \frac{1}{|\log\varepsilon|}\int_{\Omega}(\nabla u_\varepsilon\cdot \mathbf{X})\Big(\Delta u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2)\Big) = 0.\end{aligned} \tag{47}$$

In view of the last formula, we will analyze the weak limit of the stress-energy tensor

$$\alpha_\varepsilon = \left(Id - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)}\right) d\mu_\varepsilon.$$

Clearly, $|\alpha_\varepsilon| \le CN\mu_\varepsilon$, and we may assume that

$$\alpha_\varepsilon \rightharpoonup \alpha_* \equiv A\cdot \mu_*,$$

where $A$ is an $N\times N$ symmetric matrix. Since the symmetric matrix $\nabla u_\varepsilon \otimes \nabla u_\varepsilon$ is non-negative, we have

$$A \le Id. \tag{48}$$

On the other hand,

$$Tr(e_\varepsilon(u_\varepsilon)\,Id - \nabla u_\varepsilon \otimes \nabla u_\varepsilon) = (N-2)e_\varepsilon(u_\varepsilon) + 2V_\varepsilon(u_\varepsilon).$$

Therefore, since the trace is a linear operation, passing to the limit we obtain, using Theorem 35,

$$Tr(A) = (N-2). \tag{49}$$

Passing to the limit in (47), and using the decomposition in Theorem 34 (see also Remark 2, we obtain

$$\int_\Omega A^{ij}\frac{\partial X^i}{\partial x_j}\,d(\mu_* \llcorner \Sigma_\mu) + \int_\Omega \left(\frac{|\nabla\Phi_*|^2}{2}\delta_{ij} - \frac{\partial\Phi_*}{\partial x_i}\frac{\partial\Phi_*}{\partial x_j}\right)\frac{\partial X^i}{\partial x_j}\,dx = 0. \tag{50}$$

On the other hand, $\Phi_*$ is harmonic,

$$-\Delta\Phi_* = 0. \tag{51}$$

Multiplying (51) by $\mathbf{X}\cdot\nabla\Phi_*$, we obtain

$$\int_\Omega \left(\frac{|\nabla\Phi_*|^2}{2}\delta_{ij} - \frac{\partial\Phi_*}{\partial x_i}\frac{\partial\Phi_*}{\partial x_j}\right)\frac{\partial X^i}{\partial x_j}\,dx = 0. \tag{52}$$

Combining (50) and (52) we have therefore proved

**Lemma 4** *For every* $\mathbf{X} \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^N)$,

$$\int_\Omega A^{ij}\frac{\partial X^i}{\partial x_j}\,d(\mu_*^t \llcorner \Sigma_\mu) = 0. \tag{53}$$

Recall that we already know that $\Sigma_\mu$ is rectifiable. Comparing (53) with (44), it is tempting to prove that the matrix $A$ corresponds to the orthogonal projection $P$ onto the tangent space $T_x\Sigma_\mu$.

**Lemma 5** *We have*

$$A(x)\left[\int_{T_x\Sigma_\mu} \nabla\chi(y)\,d\mathcal{H}^{N-2}(y)\right] = 0 \qquad \textit{for } \mathcal{H}^{N-2}\textit{-a.e. } x \in \Sigma_\mu, \tag{54}$$

*and for all* $\chi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R})$.

*Proof.* Let $x \in \Sigma_\mu$ be such that $T_x\Sigma_\mu$ exists and such that $x$ is a Lebesgue point for $\Theta_*$ (with respect to $\mathcal{H}^{N-2}$) and for $A$ (with respect to $\mu_* \llcorner \Sigma_\mu$). For $r > 0$, consider the vector field $\mathbf{X}_{r,l}(y) = \chi(\frac{x-y}{r})e_l$. Inserting $\mathbf{X}_{r,l}$ into (53) and letting $r \to 0$, we obtain the desired result.

A straightforward consequence is

**Corollary 37** *For $x$ as in Lemma 5,*

$$(T_x\Sigma_\mu)^\perp \subseteq \operatorname{Ker} A(x).$$

With a little more elementary linear algebra, we further deduce

**Corollary 38** *For $x$ as in Lemma 5, $A = P$ is the orthogonal projection onto the tangent space $T_x\Sigma_\mu$.*

*Proof.* By (48), $A \leq Id$, and therefore all the eigenvalues $\lambda_1, ..., \lambda_N$ of $A$ are less or equal to 1. By (49), $\mathrm{Tr}(A) \geq N-2$, so that $\sum_{i=1}^N \lambda_i \geq N-2$. On the other hand, by Corollary 37, $A$ has at least two eigenvalues, say $\lambda_1$ and $\lambda_2$, equal to zero. Therefore, $\lambda_i = 1$ for $i = 3, ..., N$. In particular $A$ is an orthogonal projection on an (N-2)-dimensional space. Since $\mathrm{Ker}\, A(x) \supseteq (T_x\Sigma_\mu)^\perp$, and since $\dim(T_x\Sigma_\mu) = N-2$, the conclusion follows.

Combining the previous arguments, we have finally proved

**Theorem 39** *The measure $\mu_* \llcorner \Sigma_\mu$ has a first variation and*

$$\delta(\mu_* \llcorner \Sigma_\mu) = 0,$$

*i.e. $(\Sigma_\mu, \Theta_*)$ is a stationary varifold.*

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